Math1013 Calculus IB
Sample Final Exam Solution

Part I: Multiple Choice Questions

`MC Answers:  White Version`

Question	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
Answer	b	b	c	a	c	e	b	d	b	c	a	e	b	c	e	d	a	a	d	b	c	a

Brief answers to the `white version`

1. For which constant k can the function $f(x) = \begin{cases} 4e^x + x - k & \text{if } x \le 0 \\ \dfrac{\sin kx}{x} & \text{if } x > 0 \end{cases}$ be continuous everywhere?

(a) 1 (b) 2 (c) 3 (d) 4 (e) None of the previous

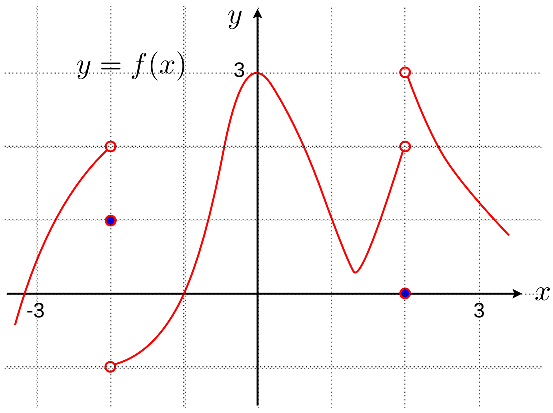
Solution

The answer is (b). f is continuous at any $x > 0$, or $x < 0$. To be continuous also at $x = 0$, need:

$$\lim_{x\to 0}(4e^x + x - k) = \lim_{x\to 0}\frac{\sin kx}{x} = k\lim_{x\to 0}\frac{\sin kx}{kx}$$

i.e., $4 - k = k$, and hence $k = 2$.

2. Find $\displaystyle\lim_{x\to -2^-}\frac{|f(|x|) - 2|}{f(x)+1}$ according to the given graph of f below.



(a) 0 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$ (e) Does not exist

Solution

The answer is (b). When $x \to -2^-$, $f(x) \to 2$, $|x| \to 2^+$, and $f(|x|) \to 3$, hence

$$\lim_{x\to -2^-}\frac{|f(|x|) - 2|}{f(x)+1} = \frac{|3-2|}{2+1} = \frac{1}{3}$$

3. Find the horizontal asymptote of the function $y = \left(\cos\dfrac{1}{2x} - \dfrac{3}{x^2}\right)\left(1 + x\sin\dfrac{1}{x}\right)$.

   (a) 0 (b) 1 (c) 2 (d) 3 (e) Does not exist

   Solution

   Then answer is (c). $\displaystyle\lim_{x\to\infty}\left(\cos\frac{1}{2x} - \frac{3}{x^2}\right)\left(1 + x\sin\frac{1}{x}\right) = \lim_{x\to\infty}\left(\cos\frac{1}{2x} - \frac{3}{x^2}\right)\lim_{x\to\infty}\left(1 + \frac{\sin\frac{1}{x}}{\frac{1}{x}}\right) =$ $\displaystyle 1 \cdot \left(1 + \lim_{x\to\infty}\frac{\cos\frac{1}{x}\cdot\frac{-1}{x^2}}{-\frac{1}{x^2}}\right) = 1 + 1 = 2$ by the L'Hôpital's rule. Note also $\displaystyle\lim_{x\to\infty}\left(\cos\frac{1}{2x} - \frac{3}{x^2}\right)\left(1 + x\sin\frac{1}{x}\right) = 2$.

4. A particle moves with position function given by $s(t) = 4t^3 - 6t^2 - 72t + 48$, for $t \geq 0$, where t is measured in seconds and s in meters. At what time is the particle at rest?

   (a) $t = 3$ sec (b) $t = 6$ sec (c) $t = 9$ sec (d) $t = 12$ sec (e) $t = 16$ sec

   Solution

   The answer is (a). The velocity is 0, i.e., $s'(t) = 12t^2 - 12t - 72 = 12(t^2 - t - 6) = 12(t-3)(t+2) = 0$, when $t = 3$ s.

5. A mass attached to a spring has position function given by $s = 4\sin(3t)$, where s is the distance of the mass from its equilibrium position in centimeters, t is time in seconds. What is the acceleration of the mass at $t = \frac{\pi}{2}$?

   (a) 12 cm/s^2 (b) 24 cm/s^2 (c) 36 cm/s^2 (d) 48 cm/s^2 (e) 60 cm/s^2

   Solution

   The answer is (c). $\displaystyle\left.\frac{d^2 s}{dt^2}\right|_{x=\frac{3\pi}{2}} = -36\sin\tfrac{3\pi}{2} = 36$ cm/s^2.

6. Find $f'(1)$ where $f(x) = \dfrac{\cos\pi x}{x + \ln x}$

   (a) -2 (b) -1 (c) $\frac{1}{2}$ (d) 1 (e) 2

   Solution

   The answer is (e). By the Quotient Rule, $f'(x) = \frac{(x+\ln x)(-\pi\sin\pi x) - \cos\pi x(1+\frac{1}{x})}{(x+\ln x)^2}$, and hence $f'(1) = 2$

7. If $\dfrac{d}{dx}\left[f(\frac{1}{3}x^3)\right] = 2x^5$, what is $f'(x)$?

(a) $3x$ (b) $6x$ (c) $3x^2$ (d) $2x^3$ (e) $\frac{2}{3}x^5$

Solution

The answer is (b). By the Chain Rule, $\dfrac{d}{dx}\left[f(\frac{1}{3}x^3)\right] = f'(\frac{1}{3}x^3) \cdot x^2 = 2x^5$; i.e., $f'(\frac{1}{3}x^3) = 2x^3 = 6 \cdot (\frac{1}{3}x^3)$, and hence $f'(x) = 6x$.

8. The increasing function $f(x) = x^3 + 4x - 2$ has an inverse function f^{-1}. Find the derivative $(f^{-1})'(-2)$.

(a) $-\frac{3}{4}$ (b) $-\frac{1}{3}$ (c) $\frac{1}{7}$ (d) $\frac{1}{4}$ (e) $\frac{1}{3}$

Solution

The answer is (d). Note that $f(0) = -2$, and $(f^{-1})'(-2) = \dfrac{1}{f'(f^{-1}(-2))} = \dfrac{1}{f'(0)} = \dfrac{1}{4}$.

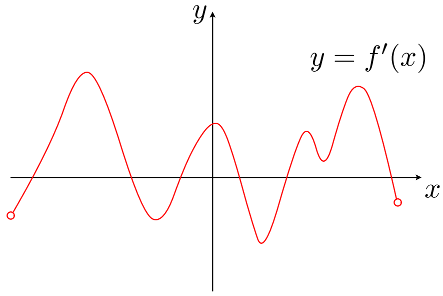
9. Find the slope of the tangent line to the curve defined by the equation $x^2(2-y) = y^3$ at the point $(1,1)$.

(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) 1 (d) 2 (e) 3

Solution

The answer is (b). By implicit differentiation, $2x(2-y) - x^2\dfrac{dy}{dx} = 3y^2\dfrac{dy}{dx}$. So at $(1,1)$, $2 - \left.\dfrac{dy}{dx}\right|_{x=1} = 2\left.\dfrac{dy}{dx}\right|_{x=1}$, and hence $\left.\dfrac{dy}{dx}\right|_{x=1} = \dfrac{1}{2}$.

10. Given the graph of the derivative function f' as shown below, exactly how many local maxima does f have?

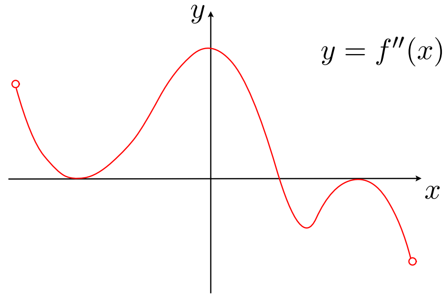


(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

Solution

The answer is (c). f' changes from positive to negative at 3 critical points on the x-axis.

11. Given the graph of the second derivative function f'' as shown below, exactly how many inflection points does f have?



(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

Solution

The answer is (a). f'' changes from positive to negative across one point on the x-axis.

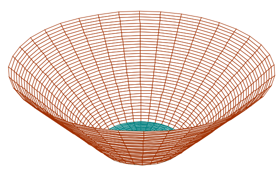
12. Find the absolute minimum of the function $y = \cos(2x) + 2x\sin(2x)$ on the interval $0 \le x \le \pi$.

(a) 1 (b) -1 (c) $-\frac{\pi}{2}$ (d) $-\frac{2\pi}{3}$ (e) $-\frac{3\pi}{2}$

Solution

The answer is (e). $f'(x) = -4x\cos 2x = 0$ if and only if $x = 0$, $x = \frac{\pi}{4}$, or $x = \frac{3\pi}{4}$ in the given interval. Easy to check that $f(\frac{3\pi}{4}) = -\dfrac{3\pi}{2}$ is the smallest function value when compare with $f(0) = 1 = f(\pi)$, and $f(\frac{\pi}{4}) = \dfrac{\pi}{2}$.

13. Water runs into a tank in the shape of a truncated regular cone with base radius 2 m, top radius 6 m and height 4 m at the rate of 10π m^3/min. How fast is the water level rising when the water is 3 m deep?



(a) 0.2 m/min (b) 0.4 m/min (c) 0.6 m/min (d) 0.8 m/min (e) 1 m/min

Solution

The answer is (b). The proportion $\frac{2}{6} = \frac{w}{w+4}$ shows that $w = 2$; i.e., the volume of the water in the tank is $V = \frac{1}{3}\pi(h+2)^3$ when the depth of the water is h m. Hence $10\pi = \dfrac{dV}{dt} = \pi(h+2)^2\dfrac{dh}{dt}$. Just plug in $h = 3$ m to find $\left.\dfrac{dh}{dt}\right|_{h=3} = 0.4$ m/min.

14. When applying the linear approximation (tangent line approximation) at $x = 0$ to approximate $f(0.02)$, where $f(x) = \sqrt{1+x} + \sin x$, the resulting approximate value is:

(a) 1.01 (b) 1.02 (c) 1.03 (d) 1.04 (e) 1.05

Solution

The answer is (c). Since $f'(x) = \frac{1}{2\sqrt{1+x}} + \cos x$, $f(0.02) \approx f(0) + f'(0)(0.02 - 0) = 1 + \frac{3}{2}(0.02) = 1.03$.

15. When applying Newton's method to find an approximate root of the equation $x^3 = 2$ with a starting value x_0, the resulting iteration formula is:

(a) $x_{n+1} = x_n - \frac{3x_n^2}{x_n^3-2}$ (b) $x_{n+1} = \frac{3x_n}{2} + \frac{3}{2x_n^2}$ (c) $x_{n+1} = \frac{2x_n}{3} - \frac{3}{2x_n^2}$
(d) $x_{n+1} = \frac{2x_n}{3} - \frac{2}{3x_n^2}$ (e) $x_{n+1} = \frac{2x_n}{3} + \frac{2}{3x_n^2}$

Solution

The answer is (e). Newton's method says $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3-2}{3x_n^2} = \frac{2x_n}{3} + \frac{2}{3x_n^2}$.

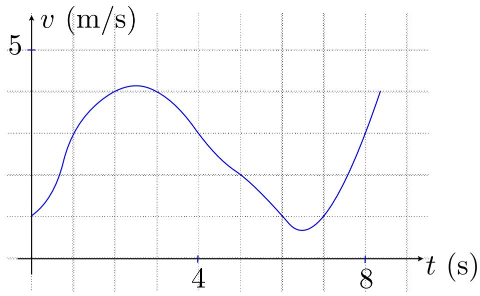
16. Find the limit: $\lim_{x\to 0}(e^x + 2x)^{\frac{1}{2x}}$.

(a) 1 (b) ∞ (c) $e^{\frac{1}{2}}$ (d) $e^{\frac{3}{2}}$ (e) $e^{\frac{1}{6}}$

Solution

The answer is (d). $\lim_{x\to 0} \ln y = \lim_{x\to 0} \dfrac{\ln(e^x + 2x)}{2x} = \lim_{x\to 0} \dfrac{\frac{e^x+2}{e^x+2x}}{2} = \dfrac{3}{2}$ by the L'Hôpital Rule.

17. The graph of the velocity function $v = v(t)$ of a travelling particle is given below. Use a midpoint Riemann sum over four subintervals of equal length to estimate the displacement of the particle over the time interval $0 \le t \le 8$, i.e., $s(8) - s(0)$ where $s = s(t)$ is the position function of the particle.



(a) 20 m (b) 22 m (c) 24 m (d) 26 m (e) 28 m

Solution

The answer is (a). $\dfrac{8}{4}[3 + 4 + 2 + 1] = 20$.

18. At time $t = 0$ minute, water starts flowing into an empty tank of volume $48\ \mathrm{m}^3$ at a rate of $r(t) = 9\sqrt{t}\ \mathrm{m}^3/\mathrm{min}$. How long will it take to fill the tank?

(a) 4 min (b) 6 min (c) 8 min (d) 10 min (e) 12 min

Solution

The answer is (a). $V' = r = 9t^{1/2}$, hence $V = 6t^{3/2} + C$, with $0 = V(0) = 0 + C$. Thus $V(t) = 6t^{3/2} = 48$ exactly when $t = 4$ min. Or solve $V(t) - V(0) = \int_0^t 9u^{1/2}du = 48$.

19. Evaluate the definite integral $\int_0^1 \sqrt{e^{3x+1}}\, dx$.

(a) $\frac{2}{3}(e^4 - e)$ (b) $\frac{3}{2}(e^4 - e)$ (c) $\frac{3}{2}(e^2 - e^{\frac{1}{2}})$ (d) $\frac{2}{3}(e^2 - e^{\frac{1}{2}})$ (e) $\frac{1}{3}(e^2 - e^{\frac{1}{2}})$

Solution

The answer is (d). Let $u = \frac{3x+1}{2}$, such that $du = \frac{3}{2}dx$ and

$$\int_0^1 \sqrt{e^{3x+1}}\, dx = \int_{1/2}^2 \frac{2}{3}e^u du = \frac{2}{3}(e^2 - e^{1/2})$$

20. Evaluate the definite integral $\int_0^2 \frac{x^2}{\sqrt{1+x^3}}dx$.

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{6}{3}$ (d) $\frac{8}{3}$ (e) 3

Solution

The answer is (b). Let $u = 1 + x^3$, such that $du = 3x^2dx$ and

$$\int_0^2 \frac{x^2}{\sqrt{1+x^3}}dx = \int_1^9 \frac{1}{3}u^{-1/2}du = \frac{2}{3}\left[u^{1/2}\right]_1^9 = \frac{2}{3}(3-1) = \frac{4}{3}$$

21. Find the limit $\lim_{n\to\infty}\left(\frac{1}{2n+1} + \frac{1}{2n+2} + \cdots + \frac{1}{2n+n}\right)$ by computing a suitable definite integral.

(a) $\ln 2$ (b) $\ln 3$ (c) $\ln\frac{3}{2}$ (d) $\frac{1}{3}$ (e) $\frac{2}{3}$

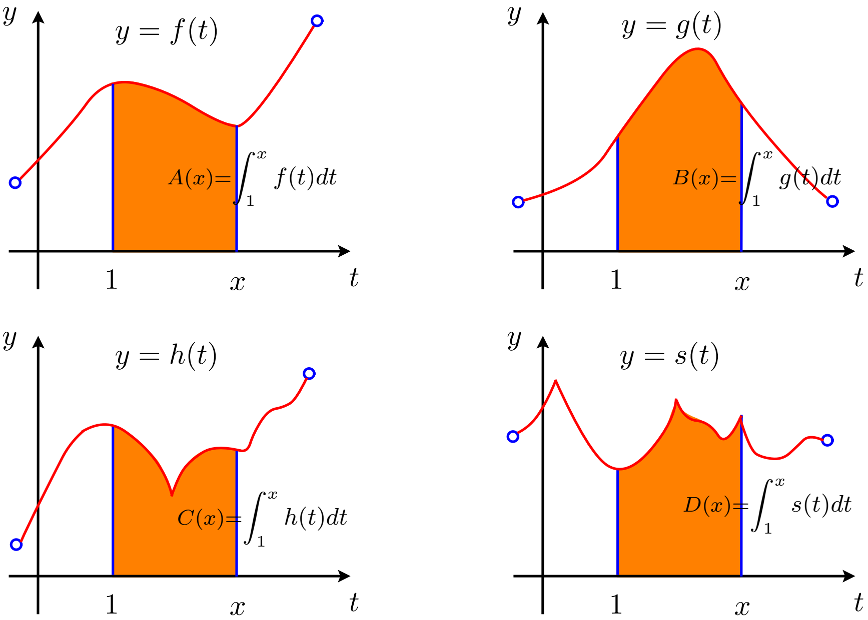
Solution

The answer is (c).

$$\lim_{n\to\infty}\left(\frac{1}{2n+1} + \frac{1}{2n+2} + \cdots + \frac{1}{2n+n}\right) = \lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{2+\frac{1}{n}} + \frac{1}{2+\frac{2}{n}} + \cdots + \frac{1}{2+\frac{n}{n}}\right)$$

$$= \int_0^1 \frac{1}{2+x}dx = \ln(2+x)\Big|_0^1 = \ln\frac{3}{2}$$

22. How many of the following area functions $A(x)$, $B(x)$, $C(x)$ and $D(x)$ for the given continuous functions as shown by the graphs below respectively can be non-differentiable at some x in the domain?



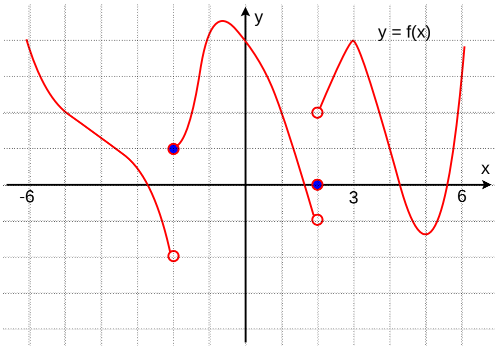
(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Solution

The answer is (a). By the Fundamental Theorem of Calculus, $A' = f$, $B' = g$, $C' = h$, $D' = s$.

Part II: Answer each of the following questions.

23. [*12 pts*] The graph of a function f is given as below, with points of discontinuity at only $x = -2$ and $x = 2$.



Determine the following limits.

(a) $\displaystyle\lim_{x\to 2^+} \frac{[f(x)]^2 - 3f(x) + 2}{[f(x)]^2 - 4}$ [4 pts]

Solution

From the graph of f, $\displaystyle\lim_{x\to 2^+} f(x)$ exists and is 2.

$$\begin{aligned}\lim_{x\to 2^+} \frac{[f(x)]^2 - 3f(x) + 2}{[f(x)]^2 - 4} &= \lim_{x\to 2^+} \frac{(f(x)-2)(f(x)-1)}{(f(x)-2)(f(x)+2)} \\ &= \lim_{x\to 2^+} \frac{(f(x)-1)}{(f(x)+2)} = \frac{2-1}{2+2} = \frac{1}{4}\end{aligned}$$

(b) $\displaystyle\lim_{x\to 2} f^{-1}(x)$, where f^{-1} is the inverse function of f when restricted to the domain $-6 < x < -2$. [4 pts]

Solution

Since f is continuous (when restricted to $(-6,-2)$), so is f^{-1}. Hence as $f(-5) = 2$,

$$\lim_{x\to 2} f^{-1}(x) = -5$$

(c) $\displaystyle\lim_{x\to 0} \frac{\int_0^{x^2} (t+1)f(t)\,dt}{3x^2}$ [4 pts]

Solution

By the L'Hopital's rule,

$$\begin{aligned}\lim_{x\to 0} \frac{\int_0^{x^2} (t+1)f(t)dt}{3x^2} &= \lim_{x\to 0} \frac{(x^2+1)f(x^2)2x}{6x} \\ &= \lim_{x\to 0} \frac{1}{3}(x^2+1)f(x^2) \\ &= \frac{1}{3}f(0) = \frac{4}{3}\end{aligned}$$

24. [10 *pts*] Consider the polynomial function $f(x) = 5x^5 - 50x^3 + 1600$.

(a) Find all intervals of increase or decrease of f. (**No partial credit.**) [3 pts]

Solution

$f'(x) = 25x^4 - 150x^2 = 25x^2(x^2 - 6) = 25x^2(x - \sqrt{6})(x + \sqrt{6})$.
Note that $f'(x) > 0$ if and only if $x < -\sqrt{6}$, or $x > \sqrt{6}$.

Interval(s) of increase: $\underline{(-\infty, -\sqrt{6}),\ (\sqrt{6}, \infty)}$

Interval(s) of decrease: $\underline{(-\sqrt{6}, \sqrt{6})}$

(b) Find all concave up intervals of f. (**No partial credit.**) [3 pts]

Solution

$f''(x) = 100x^3 - 300x = 100x(x^2 - 3) = 100x(x - \sqrt{3})(x + \sqrt{3})$, and hence $f''(x) > 0$ if and only if $-\sqrt{3} < x < 0$ or $x > \sqrt{3}$.

Concave up interval(s) : $\underline{(-\sqrt{3}, 0),\ (\sqrt{3}, \infty)}$

(c) How many real roots does f have? Justify your answer for full credit. [4 pts]

Solution

f has one real root.
In fact, since $f(-100) < 0$, and $f(\sqrt{6}) = -5 \cdot 36\sqrt{6} + 50 \cdot 6\sqrt{6} + 1600 > 0$, the continuous function f, which is increasing in the interval $(-\infty, -\sqrt{6})$, must has exactly one root lying between -10 and $-\sqrt{6}$ by the Intermediate Value Theorem.
Note also that $f(\sqrt{6}) = 5 \cdot 36\sqrt{6} - 50 \cdot 6\sqrt{6} + 1600 > 0$ is the absolute minimum of f in the interval $(-\sqrt{6}, \infty)$, by the decreasing or increasing property of f. Thus f has no other root in $(-\sqrt{6}, \infty)$.

25. ([*12 pts*]) A function and its derivative are known as

$$f(x) = \frac{3 - x^2 + x^3}{x^3}, \qquad f'(x) = \frac{x^2 - 9}{x^4}$$

Using appropriate scales on the axes in the given grid, sketch the following:

(a) All vertical and horizontal asymptotes of the function. [3 pts]

(b) All points on the graph of f with horizontal tangent line. [2 point]

(c) The graph of f, with all inflection point(s) clearly indicated together with their coordinates. [7 pts]

Solution

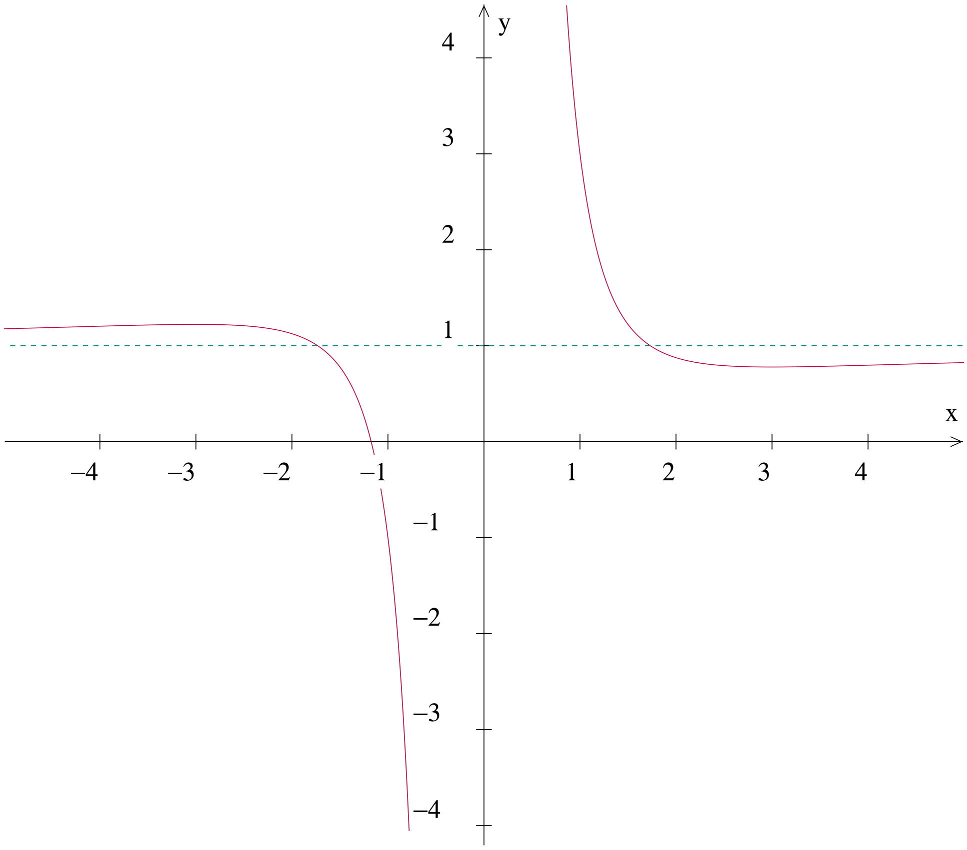
(a) Vertical asymptote: $x = 0$

Horizontal asymptote: $y = 1$, since $\lim_{x\to\infty} \frac{3 - x^2 + x^3}{x^3} = 1 = \lim_{x\to-\infty} \frac{3 - x^2 + x^3}{x^3} = 1$

(b) $f'(x) = 0$ if and only if $x = \pm 3$, where $f(-3) = \frac{11}{9}$, $f(3) = \frac{7}{9}$. Horizontal tangent line at: $(-311/9)$ and $(3, 7/9)$.

(c) $f''(x) = \frac{-2(x^2 - 18)}{x^5}$, with inflection points at $x = \pm\sqrt{18} = \pm 3\sqrt{2}$.

Inflection points: $(-3\sqrt{2}, \frac{18\sqrt{2}+5}{18\sqrt{2}})$, $(3\sqrt{2}, \frac{18\sqrt{2}-5}{18\sqrt{2}})$,



26. ([*12 pts*]) Three functions f, g, and h with continuous derivatives are given. Some function values of these functions are shown in the following table:

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$	$h(x)$	$h'(x)$
0	2	0	2	-3	1	4
2	0	3	0	-1	1	2
4	1	6	1	1	2	3

Answer the following questions:

(a) Evaluate the definite integral.

(i) $\displaystyle\int_0^2 \left[3f'(x) - 4xg'(x^2)\right]dx$ [4 pts]

$=$

Solution

$$\int_0^2 [3f'(x) - 4xg'(x^2)]dx = \Big[3f(x) - 2g(x^2)\Big]_0^2$$

$$= 3(f(2) - f(0) - 2(g(4) - g(0)) = 3(0-2) - 2(1-2) = -4$$

(ii) $\displaystyle\int_{-1}^1 [f(x+3)]^3 f'(x+3)dx$ [4 pts]

Solution

Let $u = f(x+3)$ such that $du = f'(x+3)dx$. Note also that $u = f(2) = 0$ when $x = -1$, and $u = f(4) = 1$ when $x = 1$.

$$\int_{-1}^1 [f(x+3)]^3 f'(x+3)dx = \int_0^1 u^3 du$$

$$= \frac{1}{4}\Big[u^4\Big]_0^1 = \frac{1}{4}(1-0) = \frac{1}{4}$$

(b) The graph of the function $y = 2f(x) - g(x) + h(x)$ must have a horizontal tangent line. [4 pts]

The statement above is : **True** **False** (Circle your answer.)

Brief reason:

Consider the differentiable function $F(x) = 2f(x) - g(x) + h(x)$. Then $F(0) = 4 - 2 + 1 = 3$, $F(4) = 2 - 1 + 2 = 3$. By the Mean Value Theorem, there is a c in the interval $(0, 4)$ such that

$$F'(c) = \frac{F(4) - F(0)}{4 - 0} = \frac{3-3}{4} = 0$$

i.e., $y = 2f(x) - g(x) + h(x)$ has a horizontal tangent line when $x = c$.

Or, equivalently, use the Rolle's Theorem.

27. [*10 pts*] A track field is to be built in the shape of a region bounded between two parallel lines and two semi-circles.

(a) If the perimeter of the field is exactly 4 km, find a design to maximize the rectangular area within it. In particular, what are the side lengths of the rectangular area of your design, and what is the maximum rectangular area thus obtained? [4 points]

Solution

Let the width of the rectangular be x and height be y. Then $2x + \pi y = 4$, and the rectangular area

$$A = xy = \frac{1}{\pi}x(4 - 2x), \qquad (0 < x < 2)$$

$$\frac{dA}{dx} = \frac{1}{\pi}(4 - 4x) = 0 \Longleftrightarrow x = 1$$

Noting that $\dfrac{d^2A}{dx^2} = -\dfrac{2}{\pi} < 0$, the function is concave down for $0 < x < 2$. Hence an absolute maximum is reached at $x = 1$ m, $y = \frac{2}{\pi}$ m, with maximum area

$$A_{\text{max}} = \frac{2}{\pi}$$

(Or, use the first derivative test.)

(b) If you want to keep the maximum rectangular area obtained in part (a) in a track field of the required shape above, what is the smallest possible perimeter? [6 points]

Solution

If $A = \frac{2}{\pi} = xy$, the perimeter function is

$$p = 2x + \pi y = 2x + \frac{2}{x}, \qquad (0 < x < \infty)$$

$$\frac{dp}{dx} = 2 - \frac{2}{x^2} = \frac{2(x^2 - 1)}{x^2}$$

Hence the critical point of the perimeter function is $x = 1$.

Either by the first derivative test, or by the concave up property from $\dfrac{d^2p}{dx^2} = \dfrac{4}{x^3} > 0$ for $x > 0$, a minimum is reached at $x = 1$. The smallest possible perimeter is

$$p_{\text{min}} = 2 + 2 = 4$$

(Remark: Well, if you know the *geometric inequality* $\sqrt{ab} \leq \frac{a+b}{2}$ well, both problems could be solved without using derivatives.)