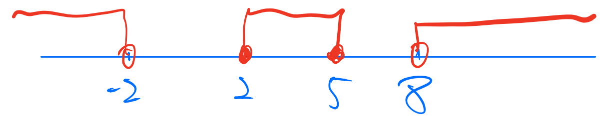
$ab \geq 0 \iff a \geq 0,\ b \geq 0$ or $a \leq 0,\ b \leq 0$ $\qquad \frac{a}{b} \geq 0$

Exercise Solve the inequality $\dfrac{(x-2)(x-5)}{(x+2)(x-8)} \geq 0.$

(Note that the four numbers $-2, 2, 5,$ and 8 divide the real line into five disjoint open intervals. Do the sign checking for each of these intervals.)

$\dfrac{B \cdot C}{A \cdot D} \geq 0$

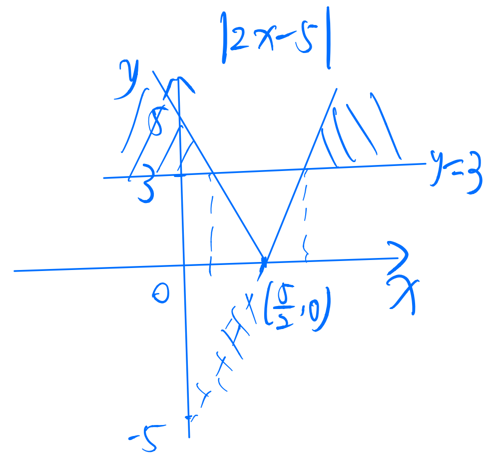


$A > B > C > D \geq 0 \qquad x > 8 \qquad x \neq -2,\ x \neq 8$

$0 \geq A > B > C > D \qquad x < -2 \qquad (-\infty, -2) \cup [2, 5] \cup (8, +\infty)$

$A > B \geq 0 \geq C > D \qquad 2 \leq x \leq 5$

(ii)

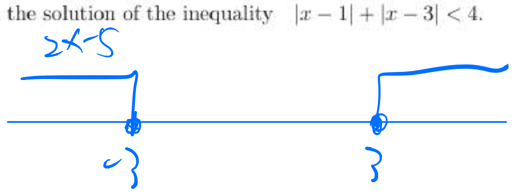


Exercise Find the solution of the inequality $|2x-5| \geq 3$ by
(i) using a sign line;
(ii) sketching the graph of $y = |2x-5|$.

Exercise Find the solution of the inequality $\left|3 - \frac{5}{x}\right| \geq 1$

Exercise Find the solution of the inequality $|x-1| + |x-3| < 4$.

(i)



$x \geq 3, \quad |x-3| = x-3, \quad |x-1| = x-1. \qquad 3 \leq x < 4$

$x-1+x-3 < 4$

$2x < 8$

$x < 4$

$1 \leq x < 3, \quad |x-1| = x-1, \quad |x-3| = 3-x \qquad 1 \leq x < 3 \qquad 0 < x < 4$

$x-1+3-x < 4$

$2 < 4$ ✓

$x < 1, \quad |x-1| = 1-x, \quad |x-3| = 3-x \qquad 0 < x < 1$

$1-x+3-x < 4$

$0 < x$

Remark Just to recall a few basic properties of absolute values:

1. $|-a| = |a|$
2. $|ab| = |a||b|$
3. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$
4. $|a+b| \leq |a| + |b|$ *(Triangle Inequality)*

   where equality holds if and only if a, b are of the same sign (equivalently $ab > 0$), or one of them is 0.

The triangle inequality follows easily from

$$\begin{aligned} |a+b|^2 &= (a+b)^2 = a^2 + 2ab + b^2 \\ &= |a|^2 + 2ab + |b|^2 \\ &\leq |a|^2 + 2|a||b| + |b|^2 = (|a| + |b|)^2 \\ |a+b| &\leq |a| + |b| \end{aligned}$$

where equality holds if and only if $ab = |ab|$, or equivalently, $ab \geq 0$.

Some Elementary Functions

Some elementary mathematical functions you need to get familiar with in Math1013 are:

- Polynomial functions; e.g., $f(x) = x^3 + 2x^2 - 4x + 5$.
- Rational functions; e.g., $f(x) = \dfrac{x^3 + 2x^2 - 4x + 5}{x^2 + 2x + 7}$.
- Power functions; e.g., $f(x) = x^{3/2}$.
- Exponential functions; e.g., $f(x) = 10^x$.
- Logarithmic functions; e.g., $f(x) = \log_{10} x$.
- Trigonometric functions; e.g., $\sin x$, $\cos x$, $\tan x$.
- Inverse trigonometric functions; e.g., $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$.

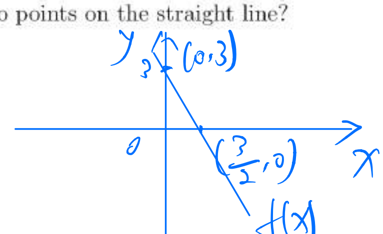
Exercise Review the basic things about "*linear functions*", i.e., functions of the form $y = mx + c$, where $m \neq 0$ is a constant called the *slope* of the function, and c a constant called the *y-intercept*.

(i) How do you find the "*x-intercept*", "*y-intercept*" and *slope* of a linear function? For example, determine the intercepts of the linear function $f(x) = -2x + 3$ and sketch its graph.

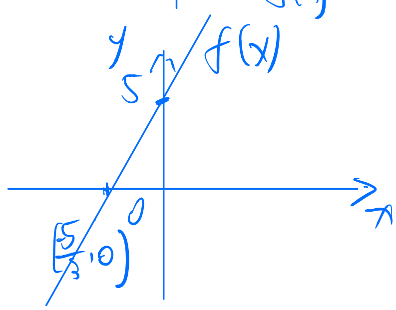
(ii) Sketch the graph of a few more linear functions; e.g., $y = 3x + 5$, or $y = -2x - 6$.

(iii) What sorts of given conditions are sufficient for you to figure out the equation of a straight line? For example, what if (i) $(2, 3)$ is a point on the straight line, and 2 is the slope; or (ii) $(2, 3)$ and $(6, 9)$ are two points on the straight line?

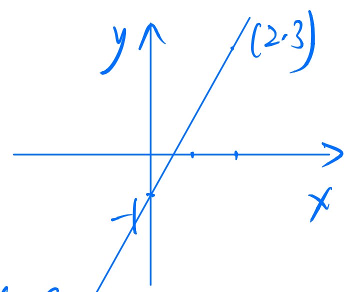
(i)



(ii)



(ii)



$y = 2x + b$

$3 = 2 \cdot 2 + b$

$b = -1$

$y = 2x + 1$

$y = kx + c$

$\begin{cases} 9 = 6k + c \\ 3 = 2k + c \end{cases} \Rightarrow \begin{cases} k = \\ c = \end{cases}$

Basic Operations on Functions

Given real-valued functions f and g, we can define new functions $f+g$ (*sum*), fg (*product*), and $\dfrac{f}{g}$ (*quotient*) simply by setting the following rules:

$$(f+g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

as long as both function values, $f(x)$ and $g(x)$, are well-defined, and the corresponding arithmetic operations on them are valid.

The main issue is that we need to be careful with the domains of these functions.

- For either $(f+g)(x)$ or $(fg)(x)$, the input value x must be in both the domain of f and the domain of g in order to have well-defined function values to add or to multiply. Hence the domain of $f+g$, or fg, is

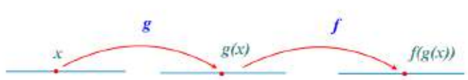
$$\{x : x \text{ is is the domain of } f \text{ and } x \text{ is also in the domain of } g\}$$

- For $\frac{f(x)}{g(x)}$ to be well-defined, $f(x)$ and $g(x)$ have to be well-defined, and $g(x)$ has to be non-zero. Hence the domain of the function $\frac{f}{g}$ is

$$\{x : x \text{ is is the domain of } f, \text{ and } x \text{ is in the domain of } g, \text{ and } g(x) \neq 0\}$$

In addition to arithmetic operations, one can also connect two "input-output machines" (functions) to form a new function, called the *composition* of f and g and denoted by the notation $f \circ g$, which is defined by

$$(f \circ g)(x) = f(g(x))$$



Obviously, we need $g(x)$ to be well-defined first, and then $g(x)$ to be in the domain of f in order to have a well-defined function value $f(g(x))$. Hence the domain of $f \circ g$ is given by

$$\text{domain of } f \circ g = \{x : x \text{ is in the domain of } g \text{ and } g(x) \text{ is in the domain of } f\}$$

Example 3. Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{x+1}{x-2}$. Find $f \circ f$, $f \circ g$ and $g \circ f$.

$$(f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} \stackrel{?}{=} x$$

$\frac{1}{\frac{1}{x}} \stackrel{?}{=} x$ $\quad$ $\frac{1}{x} \neq 0$. $x \neq 0$

$(f \circ f)(x)$

D: $x \neq 0$.

$(f \circ g)(x) = f(g(x)) = f\left(\frac{x+1}{x-2}\right) = \frac{1}{\frac{x+1}{x-2}}$

$x - 2 \neq 0$

$\frac{x+1}{x-2} \neq 0$

$x \neq 2$, $x \neq -1$.

$(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{\frac{1}{x}+1}{\frac{1}{x}-2}$

$x \neq 0$

$\frac{1}{x} - 2 \neq 0$

$x \neq 0$, $x \neq \frac{1}{2}$.

Exercise. Consider $f(x) = 2x - 1$, $g(x) = \frac{x^2+1}{x}$. Find the following functions, and determine their domains.

(a) $f(g(x))$ $\quad$ (b) $g(f(x))$ $\quad$ (c) $(f \circ g \circ f)(x) = f(g(f(x)))$

D:

(a) $f(g(x)) = 2 \cdot \frac{x^2+1}{x} - 1 \left(= \frac{2(x^2+1) - x}{x} \right)$ $\quad$ $x \neq 0$.

(b) $g(f(x)) = \frac{(2x-1)^2+1}{2x-1}$ $\quad$ $2x - 1 \neq 0$, $x \neq \frac{1}{2}$.

(c) $f(g(f(x))) = f\left(\frac{(2x-1)^2+1}{2x-1}\right)$

$= 2 \frac{(2x-1)^2+1}{2x-1} - 1$

$2x - 1 \neq 0$,

$x \neq \frac{1}{2}$

$f(-x) = \begin{cases} f(x) \\ -f(x) \\ \text{other} \end{cases}$

- **Even and Odd Functions**

A function $y = f(x)$ is called an $\begin{cases} \textit{even function} \text{ if } f(-x) = f(x) \\ \textit{odd function} \text{ if } f(-x) = -f(x) \end{cases}$ for all x in the domain of f.

$\sin x$, $\cos x$

- **Periodic Functions**

A function $f(x)$ is *periodic* if there is a number $T \neq 0$ such that

$$f(x+T) = f(x)$$

for all x in the domain. The smallest such $T > 0$, if it exists, is called the *(fundamental) period* of the periodic function.

$x_1 < x_2 \Leftrightarrow f(x_1) < f(x_2)$

- **Increasing and Decreasing Functions**

A function $y = f(x)$ is called $\begin{cases} \text{an } \textit{increasing function} \text{ if } f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \\ \text{a } \textit{decreasing function} \text{ if } f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \end{cases}$ for all x_1, x_2 in the domain of f.

Transformations of Graphs

As long as you are familiar with the graphing process, it is pretty easy to see how the graph of a function $y = f(x)$ is related to the graphs of $y = f(x) + k$, $y = f(x+k)$, $y = kf(x)$ and $y = f(kx)$ for some fixed constant k.

(i) Graph of $y = f(x) + k$: $\begin{cases} \text{upward shifting of the graph of } f \text{ by } k \text{ units if } k > 0 \\ \text{Downward shifting of the graph of } f \text{ by } k \text{ units if } k < 0 \end{cases}$

(ii) Graph of $y = f(x+k)$: $\begin{cases} \text{Shifting the graph of } f \text{ to the right by } |k| > 0 \text{ units if } k < 0 \\ \text{Shifting the graph of } f \text{ to the left by } k \text{ units if } k > 0 \end{cases}$

(iii) Graph of $y = -f(x)$: Reflecting the graph of f across the x-axis.

(iv) Graph of $y = f(-x)$: Reflecting the graph of f across the y-axis.

(v) Graph of $y = kf(x)$, where $k > 0$: $\begin{cases} \text{Stretching the graph of } f \text{ in } y\text{-direction by a factor of } k \text{ if } k > 1 \\ \text{Compressing the graph of } f \text{ in } y\text{-direction by a factor of } k \text{ if } 0 < k < 1 \end{cases}$

(vi) Graph of $y = f(kx)$, where $k > 0$: $\begin{cases} \text{Compressing the graph of } f \text{ in } x\text{-direction by a factor of } k \text{ if } k > 1 \\ \text{Stretching the graph of } f \text{ in } x\text{-direction by a factor of } k \text{ if } 0 < k < 1 \end{cases}$

Example 8. Find the inverse function $f^{-1}(x)$ for the function $f(x) = \dfrac{3x+2}{2x-1}$.

Let $y = \dfrac{3x+2}{2x-1}$. Then

$$y(2x-1) = 3x+2 \Longleftrightarrow (2y-3)x = y+2\,.$$

Hence

$$x = \frac{y+2}{2y-3} = f^{-1}(y) \quad and \quad f^{-1}(x) = \frac{x+2}{2x-3}.$$

The domain of f^{-1}, which is the range of f, is given by $x \neq \frac{3}{2}$; i.e., $(-\infty, \frac{3}{2}) \cup (\frac{3}{2}, \infty)$.
The range of f^{-1}, which is the domain of f, is given by $x \neq \frac{1}{2}$; i.e., $(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$

Exercise Find the inverses of the following functions, and their domains and ranges.

(a) $y = 4x^3 + 5$ (b) $y = \dfrac{x}{\sqrt{x^2+1}}$

(a) $x^3 = \dfrac{y-5}{4}$

$x = \sqrt[3]{\dfrac{y-5}{4}} = f^{-1}(y)$

$f^{-1}(x) = \sqrt[3]{\dfrac{x-5}{4}}$ domain: $x \in (-\infty, \infty)$

range: $f^{-1} \in (-\infty, \infty)$

(b) $y = \dfrac{x}{\sqrt{x^2+1}}$

$y^2 x^2 + y^2 = x^2$

$x^2 = \dfrac{y^2}{1-y^2}$

$x = \pm\sqrt{\dfrac{y^2}{1-y^2}} = f^{-1}(y)$

$f^{-1}(x) = \pm\sqrt{\dfrac{x^2}{1-x^2}}$. domain $(-1, 1)$

range: $(-\infty, \infty)$

- Note that we have the following identities:

$$\begin{array}{lll} \text{(i)} & \sin^2\theta + \cos^2\theta = 1 & \text{(Pythagoras Theorem!)} \\ \text{(ii)} & \cos\theta = \sin(\theta + \frac{\pi}{2}) & \text{(Graph shifting!)} \\ \text{(iii)} & \sin\theta = \cos(\theta - \frac{\pi}{2}) & \text{(Graph shifting!)} \end{array}$$

In addition to the identity $\sin^2\theta + \cos^2\theta = 1$, we have the following identities:

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

For example,

$$1 + \tan^2\theta = 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta$$

To derive the angle sum formula $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$, just consider the following figure:

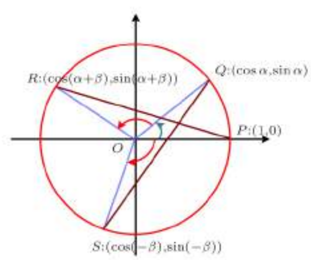
△ slides.

Note that the two triangles $\triangle POR$ and $\triangle SOQ$ are congruent, as you can rotate one to the other by an angle of β. In particular, we have

$$PR = SQ, \qquad \text{or} \qquad PR^2 = SQ^2$$

Recall that the distance between two points (x_1, y_1) and (x_2, y_2) is given by

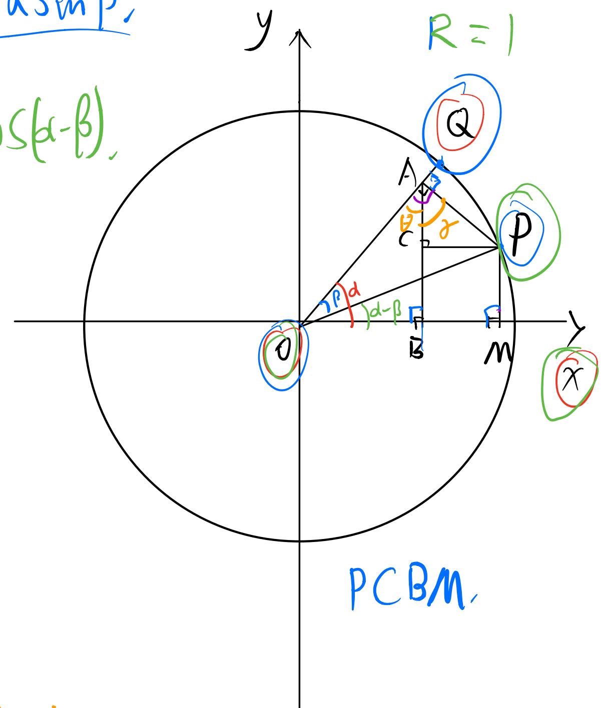
$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



The identity follows then from $PR^2 = SQ^2$:

$$\begin{aligned} (\cos(\alpha+\beta) - 1)^2 + (\sin(\alpha+\beta) - 0)^2 &= (\cos\alpha - \cos(-\beta))^2 + (\sin\alpha - \sin(-\beta))^2 \\ \cos^2(\alpha+\beta) - 2\cos(\alpha+\beta) + 1 + \sin^2(\alpha+\beta) &= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta \\ &\quad + \sin^2\alpha + 2\sin\alpha\sin\beta + \sin^2\beta \\ 2 - 2\cos(\alpha+\beta) &= 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta \\ \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{aligned}$$

$$\cos(\alpha-\beta) = \cos\alpha\cdot\cos\beta + \sin\alpha\sin\beta.$$



$$|OM| = |OP|\cdot\cos(\alpha-\beta) = \cos(\alpha-\beta).$$

$$|OM| = |OB| + |BM| \quad \cdots \quad (1)$$

$$\begin{cases} |OB| = |OA|\cos\alpha \\ |OA| = |OP|\cdot\cos\beta = \cos\beta \end{cases}$$

$$\begin{cases} |BM| = |PC| = |PA|\cdot\sin\alpha \\ \gamma+\theta = 90^\circ = \alpha+\theta \Rightarrow \gamma = \alpha. \\ |PA| = |OP|\cdot\sin\beta = \sin\beta \end{cases}$$

$$|OM| = |OB| + |BM| = \cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta$$
$$= \cos(\alpha-\beta)$$

Trigonometric Identities

Angle addition and subtraction

Δ $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$ ①

$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$ ②

$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

$\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

$\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$

Product to sum and sum to product

$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$ Δ ✓

$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$

$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$

$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$

$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$ ✓.

$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

All these formulas can be derived from one identity

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\beta\sin\alpha$$

$$\sin(\alpha+\beta) = \cos\left(\tfrac{\pi}{2} - (\alpha+\beta)\right)$$

$$\cos\left(\left(\tfrac{\pi}{2}-\alpha\right)+\beta\right) = \cos\left(\tfrac{\pi}{2}-\alpha\right)\cos\beta - \sin\left(\tfrac{\pi}{2}-\alpha\right)\sin\beta$$

$$= \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

$$\sin 2\alpha = \sin(\alpha+\alpha) = \sin\alpha\cos\alpha + \sin\alpha\cos\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = \uparrow \quad \uparrow$$

$$(1 = \cos^2\alpha + \sin^2\alpha)$$

① + ②: $\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\cdot\sin\alpha\cos\beta + 0.$

$$\sin\alpha = \sin\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right)$$

$$= \sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} + \cancel{\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}$$

$$\sin\beta = \sin\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)$$

$$= \sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} - \cancel{\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}$$

$$\sin\alpha + \sin\beta = 2\cdot\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$\cos^2\alpha = 1 - \sin^2\alpha$

$\cos 2\alpha = 1 - 2\sin^2\alpha$

$\sin 2\alpha = 2\cdot\sin\alpha\cdot\cos\alpha$

Exercise.

1. Work out the triple angle formulas for $\sin 3\alpha$, $\cos 3\alpha$.
   (Hint: $\sin 3\alpha = \sin(\alpha + 2\alpha)$)
2. Can you rewrite functions like $y = a\sin\omega t + b\cos\omega t$ into the form $y = R\sin(\omega t + C)$ for some constants R, ω, C?

   γ Δ

   Example. Since $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, we have

$$y = \sin t + \cos t = \sqrt{2}(\sin t \cos\frac{\pi}{4} + \cos t \sin\frac{\pi}{4}) = \sqrt{2}\sin(t + \frac{\pi}{4}) .$$

$\frac{\pi}{4} = 45^\circ$.

Hint: consider

$$a\sin\omega t + b\cos\omega t = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin\omega t + \frac{b}{\sqrt{a^2+b^2}}\cos\omega t\right]$$

which can be rewritten as $R\sin(\omega t + C)$, or $R\cos(\omega t + C)$ for suitable choice of C.

1. $\sin 3\alpha = \sin(2\alpha + \alpha)$
$= \sin 2\alpha \cos\alpha + \cos 2\alpha \sin\alpha$
$= 2\sin\alpha\cdot\cos^2\alpha + (1 - 2\sin^2\alpha)\cdot\sin\alpha$
$= 2\sin\alpha\cdot(1 - \sin^2\alpha) + (1 - 2\sin^2\alpha)\sin\alpha$
$= 3\sin\alpha - 4\sin^3\alpha$

2. denote $R = \sqrt{a^2+b^2}$
$\theta = \sin^{-1}\frac{a}{R}$.

The slope of a tangent line

slope at. $(a, f(a))$ (or (a, a^n))
of $f(x) = y = x^n$

Example (tangent line problem of cubic function)

Find the equation of the tangent line to the curve defined by the equation $y = x^3$ at the point $(2, 8)$.

Consider the slope of the secant line passing through the point $(2, 8)$ and a nearby point $(2+h, (2+h)^3)$ on the curve. Then

$$m_{\sec} = \frac{(2+h)^3 - 8}{(2+h) - 2} = \frac{12h + 6h^2 + h^3}{h} = 12 + 6h + h^2 \longrightarrow 12 \quad \text{as } h \to 0$$

Using the limit notation, we have

$$\lim_{h \to 0} \frac{(2+h)^3 - 8}{(2+h) - 2} = \lim_{h \to 0} \frac{12h + 6h^2 + h^3}{h} = \lim_{h \to 0} (12 + 6h + h^2) = 12$$

Hence the slope of the tangent line is 12, and the equation of the tangent line to the graph of $y = x^3$ at $(2, 8)$ is

$$\frac{y - 8}{x - 2} = 12 \Longleftrightarrow y = 12x - 16$$

$\binom{n}{0} = \frac{n!}{n!} = 1.$

$(a+h)^n$ (binomial Thm)

$= \binom{n}{0} a^n h^0 + \binom{n}{1} a^{n-1} h^1 + \cdots$

$\cdots + \binom{n}{n-1} a^1 h^{n-1} + \binom{n}{n} a^0 h^n$

$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$

$\boxed{h \to 0}$

$$m_{sec} = \frac{(a+h)^n - a^n}{(a+h) - a} = \frac{\binom{n}{0} a^n h^0 + \binom{n}{1} a^{n-1} h^1 + \binom{n}{2} a^{n-2} h^2 + \cdots - a^n}{h}$$

$$= \frac{0 + n \cdot a^{n-1} \cdot h + \binom{n}{2} a^{n-2} \cdot h^2 + \cdots}{h} \quad /h \quad /h$$

$$= n \cdot a^{n-1} + \binom{n}{2} a^{n-2} \cdot \underset{\to 0}{h} + \cdots \to 0$$

$$\overset{h \to 0}{=} n \cdot a^{n-1}$$

Slope at (a, a^n) of $f(x) = x^n$.

Example 1. Find the equation of the tangent line to the graph of the function $y = f(x) = 2x^2 - 3$ at the point $(1, -1)$. Find also the derivative function $f'(x)$.

$\log_e x = \ln x$

slope at $(a, f(a))$ $(a, \ln a)$.

of $f(x) = \ln x$.

$f'(x) = \frac{1}{x}$

Suppose that the slope of the tangent line to the graph of $y = \log_e x$ at the point $(1, 0)$ is 1. We regard e as an unknown number we want to find.

Then the trending behavior of the slope of the secant line passing through the point $(1, 0)$ and a nearby point $(1 + h, \log_e(1 + h))$ on the graph as $h \to 0$ should be

$$m_{\text{sec}} = \frac{\log_e(1+h) - 0}{(1+h) - 1} = \frac{1}{h}\log_e(1+h)$$
$$= \log_e(1+h)^{\frac{1}{h}} \longrightarrow 1 \quad \text{as } h \to 0$$

Using the limit notation, e is the number which satisfies

$$\lim_{h\to 0} \log_e(1+h)^{\frac{1}{h}} = 1.$$

Since we have $\log_e e = 1$, one way to define the number e is

$$\boxed{e = \lim_{h\to 0}(1+h)^{\frac{1}{h}}.}$$

$\frac{h}{a} =$ "h"

Let $h = 10^{-10}$, then $(1 + 10^{-10})^{10^{10}} = 2.7182821 \cdots \approx e = 2.7182818 \cdots$

$$m_{\sec} = \frac{\ln(a+h) - \ln a}{(a+h) - a} = \frac{1}{h} \cdot \ln\left[\frac{a+h}{a}\right] = \ln\left(\frac{a+h}{a}\right)^{\frac{1}{h}} \quad (h \to 0).$$

$$\lim_{h\to 0} \ln\left(1 + \frac{h}{a}\right)^{\frac{1}{h}} = \lim_{h\to 0} \ln\left[\left(1 + \frac{h}{a}\right)^{\frac{a}{h}}\right]^{\frac{1}{a}} \quad \left(\lim_{h\to 0}\left(1+\frac{h}{a}\right)^{\frac{a}{h}} = e\right).$$

$$= \ln e^{\frac{1}{a}}$$

$$= \frac{1}{a}$$

slope at $(a, f(a))$ of $f(x) = \ln x$.

is: $\frac{1}{a}$

The phrase "$f(x)$ becomes arbitrarily close to L" means that f(x) eventually lies in the interval $(L-\varepsilon, L+\varepsilon)$, which can also be written as $|f(x)-L| < \varepsilon$.

ε. $\forall\ \varepsilon > 0$, e.g. $\varepsilon = 10^{-10}, 10^{-100}, \ldots > 0$.

The phrase "as x approaches c" refer to values of x, whose distance from c is less than some positive number δ that is, values of x within either $(c-\delta, c)$ or $(c, c+\delta)$, which can be expressed with $0 < |x-c| < \delta$.

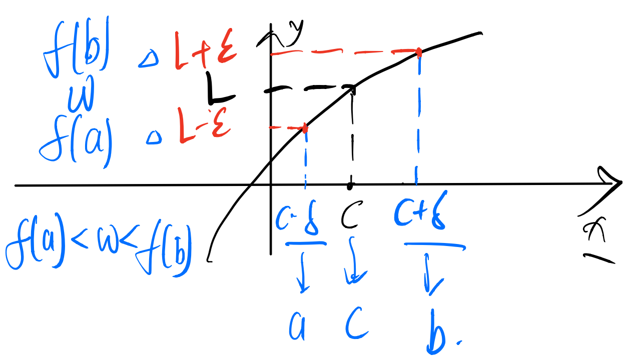
$f(c) = L$.

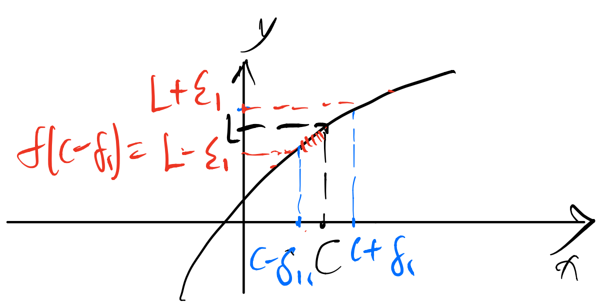
$(c-\delta, c+\delta)$. $\exists\delta,\ \delta > 0$.

The expression $\lim\limits_{x \to c} f(x) = L$ means for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x)-L| < \varepsilon$ whenever $0 < |x-c| < \delta$.

$\exists\ \delta_1 < \delta$

$\varepsilon_1 < \varepsilon$.





Continuity

We have seen that even when $f(c)$, $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ all exist, it is still possible that they are not equal.

When they are all well-defined and equal, we say that the function is *continuous* at $x = c$.

Roughly speaking, this is a mathematical way to say that no 'sudden jump' on the graph will occur when passing through $x = c$.

For most of the time in this course, we shall be dealing with the continuity of functions defined on an interval, or the union of several intervals.

- If c is a number in the domain of a function f such that a small open interval $(c-h, c+h)$ containing c, where $h > 0$, is entirely lying inside the domain of f, c is called an *interior point* of the domain of f.

  A function $y = f(x)$ is said to be *continuous at an interior point* c in its domain if $\lim_{x\to c} f(x) = f(c)$.

- If a is a number in the domain of of f which is not an interior point, then the continuity condition $\lim_{x\to a} f(x) = f(a)$ should be understood as $f(x)$ is getting closer and closer to $f(a)$ as x in the domain of f is getting closer to closer to a.

  In particular, $x \to a$ should be understood as $x \to a^+$ is a is a "left endpoint" of the domain. Similarly, $x \to a$ should be understood as $x \to b^-$ if b is a "right endpoint" of the domain.

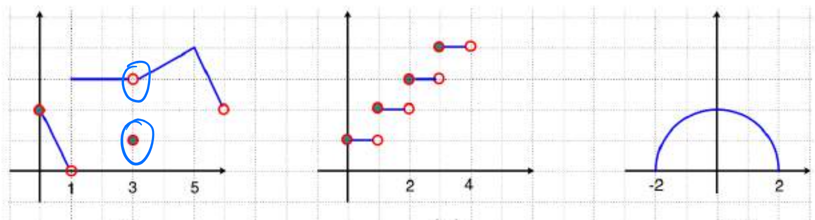
- Sometimes, d is called a *point of discontinuity* of a function f if the condition

$$\lim_{x\to a} f(x) = f(d)$$

  is not satisfied, i.e., either (i) $f(d)$ is not well-defined; or (ii) the limit does not exist at all; or (iii) $f(d)$ is well-defined but not equal to the well-defined limit $\lim_{x\to a} f(x)$.

  (According to this definition, every point not in the domain of f could be considered as a point of discontinuity of the function, which is sometime confusing.)

Example 1. It is easy to see where the following functions are continuous/discontinuous:



(i) (ii) (iii)

(i) d. $x = 1, 3, 6$ $\qquad \lim\limits_{x \to 3^-} f(x) \neq \lim\limits_{x \to 3} f(x)$

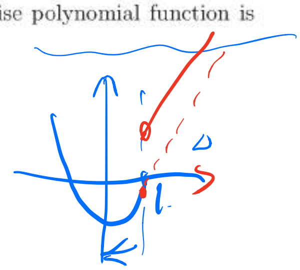
$\to 3^+$

c. $[0, 1)$ & $(1, 3)$ & $(3, 5]$ & $[5, 6)$.

(iii) in the domain $(-2, 2)$. continuous.

Example 2. Find the value of the constant k such that the following piecewise polynomial function is continuous everywhere.

$$f(x) = \begin{cases} x^2 + 3x - 2k & \text{if } x \leq 1, \\ 2x - 3k & \text{if } x > 1. \end{cases}$$



$a \neq 1$. $\lim\limits_{x \to a} f(x) = f(a)$.

$x = 1$.

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x^2 + 3x - 2k) = 4 - 2k = f(1^-)$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (2x - 3k) = 2 - 3k = f(1^+)$

$f(1^-) = 4 - 2k = 2 - 3k = f(1^+)$.

$k = -2$.

when $k = -2$. $f(x)$ is continuous everywhere.

Properties of Continuous Functions

- Sums, differences, products of continuous functions are continuous. In particular, *polynomial functions* are continuous on the entire real line.

  Recall here that a polynomial function of degree n is a function of the form

  $$f(x) = ax^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  where $a_0, a_1, \ldots, a_n$ are real numbers, and n is a non-negative integer.

- The quotient $\frac{f}{g}$ of two functions $f(x)$, $g(x)$ continuous at $x = c$ is continuous at $x = c$ as long as $g(c) \neq 0$. In particular, rational functions are continuous on the real line, except at the zeros of their denominators, i.e., continuous on their domains.

  Recall here that a *rational function* is a function of the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x), q(x)$ are polynomials with $q(x) \not\equiv 0$.

- For any positive integer n, the root function $f^{1/n}$ of a function f continuous at $x = c$ is also continuous at $x = c$, as long as the power function is well-defined on an open interval containing c.

These properties are straightforward consequences of the limit laws. For example, if f and g are continuous at a, then

$$\lim_{x \to a} f(x) = f(a), \qquad \lim_{x \to a} g(x) = g(a)$$

C C,

and hence

$$\lim_{x \to a}(f+g)(x) = \lim_{x \to a}(f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a) = (f+g)(a)$$

i.e., the function $f + g$ is also continuous at a.

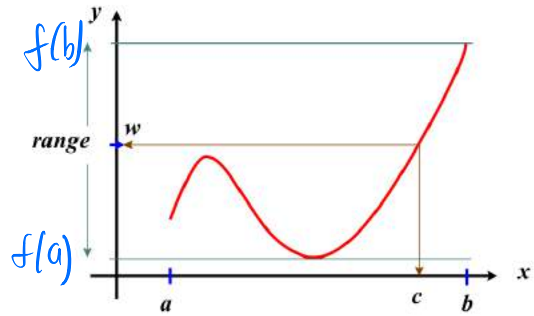
Remark. The elementary functions $\sin x$, $\cos x$, $\tan x$, a^x and $\log_a x$ are all continuous at any point in their domains. (Look at the graphs of these function!)

Remark. Note also that if f is continuous at c and g is continuous at $f(c)$, then the composition of the two functions $g \circ f$ is continuous at c. In fact, as $x \to c$, $f(x) \to f(c)$ by the continuity of f at c, and hence $g(f(x)) \to g(f(c))$ by the continuity of g at $f(c)$.

Intermediate Value Theorem

!!

Theorem (Intermediate Value Theorem). Suppose the function $y = f(x)$ is $\boxed{\text{continuous}}$ on a !! $\boxed{\text{closed}}$ interval $[a, b]$ and let w be a number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there must be a number c in (a, b) such that $f(c) = w$.



In other words, the equation $f(x) = w$ must have at least one root in the interval (a, b). The Intermediate Value Theorem is very useful in locating roots of equations.

Rolle Thm.
Lagrange mean value Thm.
Cauchy mean value Thm.

(a,b)

Example 3. Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ in the interval $(1, 2)$.

$f(x) = 4x^3 - 6x^2 + 3x - 2.$

Show that $\exists\, c \in (1,2)$. s.t. $f(c) = 0$.

$f(x)$ is continuous at $[1,2]$.

$f(1) = 4 - 6 + 3 - 2 = -1 < 0$

$f(2) = 32 - 24 + 6 - 2 = 12 > 0.$

$f(1) < 0 < f(2).$

$f(1.5) > 0.$

$c \in (1, 1.5).$

Limit Definition of Derivatives

Recall that the rate of change of a function $y = f(x)$ at $x = a$ is a certain limit called the *derivative of f at a* , which is denoted by $f'(a)$, and is defined by

$f(x) = x^n$.
$f'(x) = n x^{n-1}$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \overset{or}{=} \lim_{x\to a} \frac{f(x) - f(a)}{x - a}$$

whenever the limit exists.

$g(x) = \ln x$.
$g'(x) = \frac{1}{x}$.

- f is said to be *differentiable at $x = a$* when $f'(a)$ exists.
- Recall also that the limit $f'(a)$ can be interpreted as the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.

If we want to measure how fast the function value $y = f(x)$ changes as x varies, we consider the *derivative function* $f'(x)$, which is defined as follows:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

whenever the limit exists. Geometrically speaking, f' is the slope function of f.

Non-Differentiability

The derivative of a function may not always exist, especially if the graph of the function is not "smooth" enough; in other words, if the graph has certain "corners".

A basic example is the absolute value function $\boxed{f(x) = |x|.}$ N - D. Its derivative at $x = 0$, namely $f'(0)$, does not exist since there is no tangent line to the graph at $(0, 0)$.

More precisely, by the limit definition of derivative, we have

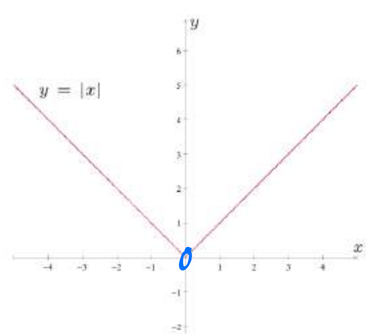
$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h| - 0}{h}$$

but

$$\lim_{h\to 0^+} \frac{|h| - 0}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

$$\lim_{h\to 0^-} \frac{|h| - 0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

i.e., the one-sided limits do not agree and therefore the limit does not exist.



Note that the "*left-hand derivative*" $f'_-(0)$ and "*right-hand derivative*" $f'_+(0)$ of $f(x) = |x|$ exist:

$$f'_-(0) = \lim_{h\to 0^-} \frac{|h|-0}{h} = -1 \qquad \text{(left-hand derivative)}$$

and

$$f'_+(0) = \lim_{h\to 0^+} \frac{|h|-0}{h} = \lim_{h\to 0+} \frac{h}{h} = 1 \qquad \text{(right-hand derivative)}$$

Basic Formulas of Derivatives

Here are the derivatives of some elementary functions. Keep in mind that these formulas are the results of some limit computations.

$$\frac{dc}{dx} = 0 \qquad \text{for any constant } c$$

$$\frac{dx^p}{dx} = px^{p-1} \qquad \text{for any constant exponent } p \neq 0$$

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\cos x}{dx} = -\sin x$$

$$\frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\cot x}{dx} = -\csc^2 x$$

$$\frac{d\sec x}{dx} = \sec x \tan x \qquad \frac{d\csc x}{dx} = -\csc x \cot x$$

$$\frac{de^x}{dx} = e^x \qquad \frac{d\ln x}{dx} = \frac{1}{x}$$

Exercise

Find (i) $\lim_{x\to -1^-} \frac{x^2-x+1}{x^2-1}$, (ii) $\lim_{x\to -1^+} \frac{x^2-x+1}{x^2-1}$. Can you find all vertical asymptotes of the function $\frac{x^2-x+1}{x^2-1}$? and any horizontal asymptotes?

Exercise

Compute the limit (a) $\lim_{x\to 0^+} \left(\frac{1}{\sqrt{x}} - \frac{1}{x}\right)$, (b) $\lim_{x\to 2} \sqrt{\frac{x^2-5x+6}{x^2-4}}$.

Exercise

Compute the limit (a) $\lim_{x\to e^2} \frac{(\ln x)^3-8}{(\ln x)^2-4}$, (b) $\lim_{x\to 0} \frac{1+\sin x}{\cos^2 x}$.

1. (i) $\lim_{x\to -1^-} \frac{x^2-x+1}{x^2-1} = \lim_{x\to -1^-} \overset{1-(-1)+1=3}{(x^2-x+1)} \lim_{x\to -1^-} \frac{1}{x^2-1}$

$= 3\cdot\infty$

$= \infty$

(ii) $\lim_{x\to -1^+} \frac{x^2-x+1}{x^2-1} = \lim_{x\to -1^+} (x^2-x+1) \lim_{x\to -1^+} \frac{1}{x^2-1}$

$= 3\cdot(-\infty)$

$= -\infty.$

$x=-1$ is vertical asymptotes

$\lim_{x\to\infty} \frac{x^2-x+1}{x^2-1}$

$f(x) = \frac{x^2-x+1}{x^2-1} = \frac{1-\frac{1}{x}+\frac{1}{x^2}}{1-\frac{1}{x^2}}$

$$\lim_{x\to\infty} \frac{1-\frac{1}{x}+\frac{1}{x^2}}{1-\frac{1}{x^2}} = \frac{1-0+0}{1-0} = \frac{1}{1} = 1.$$

$$\lim_{x\to-\infty} \frac{1-\frac{1}{x}+\frac{1}{x^2}}{1-\frac{1}{x^2}} = \frac{1}{1} = 1$$

$y=1$. is horizontal asymptote

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2. (a) $\lim_{x\to 0^+}\left(\frac{1}{\sqrt{x}}-\frac{1}{x}\right)$

$$f(x)=\frac{1}{\sqrt{x}}-\frac{1}{x}=\frac{\sqrt{x}-1}{x}\cdot\frac{\sqrt{x}+1}{\sqrt{x}+1}$$

$$=\frac{x-1}{x\cdot(\sqrt{x}+1)}.$$

$$\lim_{x\to 0^+}\frac{x-1}{x\cdot(\sqrt{x}+1)} = \lim_{x\to 0^+}(x-1)\cdot\lim_{x\to 0^+}\frac{1}{x\cdot(\sqrt{x}+1)}$$

$$= -1\cdot\infty$$

$$= -\infty.$$

(b) $\lim_{x\to 2}\sqrt{\frac{x^2-5x+6}{x^2-4}}$

$$\frac{x^2-5x+6}{x^2-4}=\frac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x+2)}.$$

$$\lim_{x\to 2}\sqrt{\frac{x-3}{x+2}} = \sqrt{\frac{-1}{4}}.$$

$(f(x)\in\mathbb{C})$ $\cancel{\mathbb{R}}$

$$=\frac{i}{2}$$

$i=\sqrt{-1}.$

3. (a) $\lim\limits_{x \to e^2} \dfrac{(\ln x)^3 - 8}{(\ln x)^2 - 4}$ denote $a = \lim\limits_{x \to e^2} \ln x$

$$\lim_{a \to 2} \frac{a^3 - 8}{a^2 - 4} = \lim_{a \to 2} \frac{(a-2)(a^2+2a+4)}{(a-2)(a+2)}$$

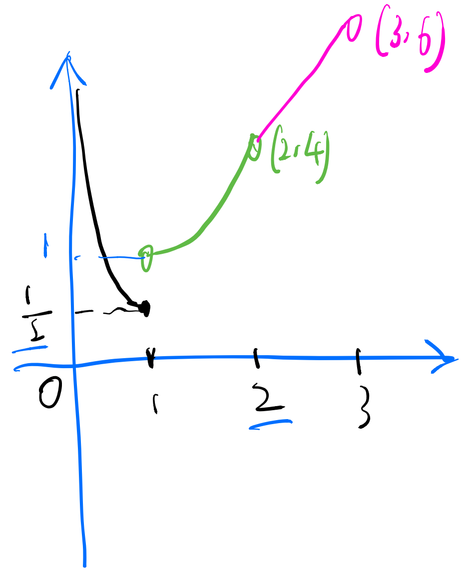
$$= \frac{4+4+4}{4} = 3.$$

(b) $\lim\limits_{x \to 0} \dfrac{1 + \sin x}{\cos^2 x} = \dfrac{1+0}{1} = 1$

Oct 5, 2020 Limits of Functions

Discuss the limits at x = 0,1,2

$$f(x)=\begin{cases}\dfrac{1}{2x}, & 0<x\le 1,\\ x^2, & 1<x<2,\\ 2x & 2<x<3,\end{cases}$$



$\lim_{x\to 0^+} f(x) = +\infty$

$\lim_{x\to 1^-} f(x) = \frac{1}{2}$

$\lim_{x\to 1^+} f(x) = 1.$

$\lim_{x\to 2^-} f(x) = 4.$ $\quad \lim_{x\to 2^+} f(x) = 4$

When x—>0, does the limit of f(x) exist?

$x\to 0^-$ $\quad x\to 0^+$

$\lim_{x\to 0} \frac{\sin x}{x} = 1.$

$$f(x)=\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}}+\frac{\sin x}{|x|}\right)。$$

$\frac{1}{x} \to -\infty$ $\quad x\to 0^-$

$e^{\frac{1}{x}}.$ $\quad \frac{1}{x}\to -\infty$

$\to 0.$

$\frac{1}{x} \to +\infty$ $\quad (x\to 0^+)$

$e^{\frac{1}{x}} \to +\infty$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left[\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}} + \frac{\sin x}{-x}\right)$

$= \frac{2+0}{1+0} - 1 = 1.$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}} + \frac{\sin x}{x}\right)$ $\quad \frac{\infty}{\infty}$

$= \quad 0+1 = 1.$

$x\to 0.$ $\quad \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$

Discuss the limits of the following functions

(1) $\lim\limits_{x\to\infty}\frac{\sin x}{x}$;

$-1 \le \sin x \le 1$

$\lim\limits_{x\to\infty}\frac{\sin x}{x}=0.$

(2) $\lim\limits_{x\to\infty} e^x \sin x$;

$\lim\limits_{x\to-\infty} e^x \sin x = 0.$

$\sin x$. asymp

$e^x \to +\infty$ $(x\to+\infty)$.

$\lim\limits_{x\to+\infty} e^x \sin x$ does not exist. ×

(3) $\lim\limits_{x\to+\infty} x^\alpha \sin\frac{1}{x}$; $=L.$

$\frac{1}{x}=t$

$x\to+\infty$ $t\to 0^+$.

$\lim\limits_{t\to 0^+}\left(\frac{1}{t}\right)^\alpha \sin t.$

$\lim\limits_{t\to 0^+}\frac{\sin t}{t}=1 \Rightarrow \alpha=1.$ $L=1.$

$\alpha<1.$ $L=0.$

$\alpha>1.$ $L=+\infty.$

(4) $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x^2}$; $(\ln x = \log_e x)$.

$\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x^2}$

$=\lim\limits_{x\to\infty} e^{x^2\ln\left(1+\frac{1}{x}\right)}$ ...... ①

$\lim\limits_{x\to-\infty} x^2\ln\left(1+\frac{1}{x}\right)$

$=\lim\limits_{x\to-\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x^2}}$ (use L'H-- Rule).

$=\lim\limits_{x\to-\infty}\frac{-\frac{1}{x(x+1)}}{-\frac{2}{x^3}}=\lim\limits_{x\to-\infty}\frac{x^2}{2(x+1)}$ (use again).

$=\lim\limits_{x\to-\infty}\frac{2x}{2}=\lim\limits_{x\to-\infty} x.$

$\lim\limits_{x\to+\infty} x^2\ln\left(1+\frac{1}{x}\right)=\lim\limits_{x\to+\infty} x.$

Squeeze Theorem

$g(x) < f(x) < h(x)$.

$$\lim_{x\to+\infty} x^{\frac{1}{x}}$$

$\exists n$,

$n \le x < n+1$

$$n^{\frac{1}{n+1}} < x^{\frac{1}{x}} < (n+1)^{\frac{1}{n}}.$$

$$\lim_{x\to-\infty} e^{x^2\ln(1+\frac{1}{x})} = e^{-\infty} = 0.$$

$$\lim_{x\to+\infty} e^{x^2\ln(1+\frac{1}{x})} = e^{+\infty} = +\infty$$

does not exist.

Prove:

$g(x) < f(x) < h(x)$.

(1) $\lim_{x\to+\infty} \frac{x^k}{a^x} = 0$

$[x]$.

$[3.67] = 3$

$[3.14] = 3$.

$x < [x]+1$

$x \ge [x]$.

$$0 < \frac{x^k}{a^x} < \frac{([x]+1)^k}{a^{[x]}}$$

$$\lim_{n\to\infty} \frac{(n+1)^k}{a^n} = 0$$

$$\Rightarrow \lim_{x\to+\infty} \frac{x^k}{a^x} = 0,$$

(2) $\lim_{x\to+\infty} \frac{\ln^k x}{x} = 0$

$\ln x = t$.

$f(x) = \frac{t^k}{e^t}$. $(a=e)$.

Similar to (1).

L'Hospital Rule (~~not~~ necessary) ✓ tool

L'Hôpital's rule states that for functions f and g which are differentiable on an open interval I except possibly at a point c contained in I, if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$, and $g'(x) \neq 0$ for all x in I with $x \neq c$, and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \boxed{\lim_{x\to c} \frac{f'(x)}{g'(x)}}.$$ easy

The differentiation of the numerator and denominator often simplifies the quotient or converts it to a limit that can be evaluated directly.

Example

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x};$$

$x \to 0$, $e^x - e^{-x} \to 1-1=0$.

$\sin x \to 0$. $\frac{0}{0}$.

$\sin' x = \cos x$ $\cos 0 \neq 0$. $=1$

$$\lim_{x\to 0} \frac{e^x - (-e^{-x})}{\cos x} = \frac{1+1}{1} = 2.$$

$$\lim_{x\to 1} \frac{x-1}{\ln x};$$

$x \to 1$ $\ln x \to 0$ $x-1 \to 0$. $\frac{0}{0}$.

$(\ln x)' = \frac{1}{x}\Big|_1$ $\frac{1}{1} = 1. \neq 0$.

$$\lim_{x\to 1} \frac{1}{\frac{1}{x}} = \lim_{x\to 1} x = 1.$$

$\frac{f(x)}{g(x)}$ $(x\to c)$, $\frac{\infty}{\infty}$

Oct 12, 2020

Rules of Differentiation

Whenever f' and g' both exist, we have the following rules:

1. $\frac{d}{dx}(af \pm bg) = a\frac{df}{dx} \pm b\frac{dg}{dx} = af' \pm bg'$ for any constants a and b.
2. **Product Rule:** $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} = fg' + gf'$
3. **Quotient Rule:** $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} = \frac{gf' - fg'}{g^2}$

Basic Formulas of Derivatives

Here are the derivatives of some elementary functions. Keep in mind that these formulas are the results of some limit computations.

$$\frac{dc}{dx} = 0 \qquad \text{for any constant } c$$

$$\frac{dx^p}{dx} = px^{p-1} \qquad \text{for any constant exponent } p \neq 0$$

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\cos x}{dx} = -\sin x$$

$$\frac{d\tan x}{dx} = \sec^2 x \qquad \frac{d\cot x}{dx} = -\csc^2 x$$

$$\frac{d\sec x}{dx} = \sec x\tan x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

$$\frac{de^x}{dx} = e^x \qquad \frac{d\ln x}{dx} = \frac{1}{x}$$

Exercise

Prove the first rule $\frac{d}{dx}(af \pm bg) = a\frac{df}{dx} \pm b\frac{dg}{dx} = af' \pm bg'$. ✓.

$$(af \pm bg)' = \lim_{h\to 0}\frac{[a\cdot f(x+h) \pm b\cdot g(x+h)] - [af(x) \pm bg(x)]}{h}$$

$$= a\cdot\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \pm b\cdot\lim_{h\to 0}\frac{g(x+h) - g(x)}{h}$$

$$= a\cdot f'(x) \pm b\cdot g'(x). \quad ✓.$$

Exercise

Prove the quotient rule: $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} = \dfrac{gf' - fg'}{g^2}$.

$$\left(\frac{f}{g}\right)' = \lim_{h\to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{f(x+h)\cdot g(x) - f(x)\,g(x+h)}{g(x+h)\cdot g(x)}}{h}$$

$$= \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}$$

$$\left(\lim_{h\to 0} \frac{1}{g(x+h)\,g(x)} = \frac{1}{g(x)^2}\right).$$

Exercise

Show that

1. $\dfrac{d\cot x}{dx} = -\csc^2 x$, ✓
2. $\dfrac{d\sec x}{dx} = \sec x \tan x$, ✓
3. $\dfrac{d\csc x}{dx} = -\csc x \cot x$. ✓

① $(\cot x)' = \left(\dfrac{\cos x}{\sin x}\right)' = \dfrac{(\cos x)'\cdot \sin x - \cos x\cdot(\sin x)'}{\sin^2 x} = \dfrac{-\sin x\cdot\sin x - \cos x\cdot\cos x}{\sin^2 x} = \dfrac{-1}{\sin^2 x}$

$= -\csc^2 x.$

② $(\sec x)' = \left(\dfrac{1}{\cos x}\right)' = \dfrac{0 - (\cos x)'\cdot 1}{\cos^2 x} = \dfrac{\sin x}{\cos^2 x} = \tan x \dfrac{1}{\cos x} = \tan x \sec x$

③ $(\csc x)' = \left(\dfrac{1}{\sin x}\right)' = \dfrac{0 - (\sin x)'\cdot 1}{\sin^2 x} = \dfrac{-\cos x}{\sin^2 x} = -\cot x \csc x$

Find the derivatives of the following functions

(1) $f(x)=3\sin x+\ln x-\sqrt{x}$; $(x^{\frac{1}{2}})' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$.

(2) $f(x)=x\cos x+x^2+3$;

(3) $f(x)=(x^2+7x-5)\sin x$;

(4) $f(x)=x^2(3\tan x+2\sec x)$;

(5) $f(x)=e^x\sin x-4\cos x+\dfrac{3}{\sqrt{x}}$;

(6) $f(x)=\dfrac{2\sin x+x-2^x}{\sqrt[3]{x^2}}$; $\dfrac{1}{\sqrt[3]{x^2}} = x^{-\frac{2}{3}}$.

(7) $f(x)=\dfrac{1}{x+\cos x}$;

(8) $f(x)=\dfrac{x\sin x-2\ln x}{\sqrt{x}+1}$;

(9) $f(x)=\dfrac{x^3+\cot x}{\ln x}$;

(10) $f(x)=\dfrac{x\sin x+\cos x}{x\sin x-\cos x}$;

(11) $f(x)=(e^x+\log_3 x)\arcsin x$;

(12) $f(x)=(\csc x-3\ln x)x^2\,\mathrm{sh}\,x$;

(13) $f(x)=\dfrac{x+\sec x}{x-\csc x}$;

(14) $f(x)=\dfrac{x+\sin x}{\arctan x}$;

(1) $f'(x) = 3\cos x + \frac{1}{x} - \frac{1}{2\sqrt{x}}$

(3) $f'(x) = (2x+7)\sin x + (x^2+7x-5)\cos x$

(2) $f'(x) = x'\cos x + x\cdot(\cos x)' + 2x + 0$
$= \cos x - x\sin x + 2x.$

(4). $f'(x) = 2x(3\tan x + 2\sec x)$
$+ x^2(3\sec^2 x + 2\tan x\sec x).$

(5) $f'(x) = (e^x)'\sin x + e^x(\sin x)' - 4(\cos x)' + 3\cdot(x^{-\frac{1}{2}})'$
$= e^x\cdot(\sin x + \cos x) + 4\sin x - \frac{3}{2}\cdot x^{-\frac{3}{2}}$

(6) $f'(x) = (2\sin x + x - 2^x)'\cdot x^{-\frac{2}{3}} + (2\sin x + x - 2^x)\cdot(x^{-\frac{2}{3}})'$
$= (2\cos x + 1 - 2^x\ln 2)\cdot x^{-\frac{2}{3}} - \frac{2}{3}(2\sin x + x - 2^x)\cdot x^{-\frac{5}{3}}$

(7). $f'(x) = -\dfrac{(x+\cos x)'}{(x+\cos x)^2} = \dfrac{\sin x - 1}{(x+\cos x)^2}$

(8). $f(x)= \dfrac{x\sin x - 2\ln x}{\sqrt{x}+1}$ $\qquad (x^{\frac{1}{2}})' = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$$f'(x) = \frac{(x\sin x-2\ln x)'(\sqrt{x}+1)-(x\sin x-2\ln x)(\sqrt{x}+1)'}{(\sqrt{x}+1)^2}$$

$$= \frac{(\sin x + x\cos x - \frac{2}{x})\cdot(\sqrt{x}+1)-(x\sin x-2\ln x)\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x}+1)^2} \qquad \frac{\cdot 2x}{\cdot 2x}$$

$$= \frac{2(x\sin x+x^2\cos x-2)(\sqrt{x}+1)-(x\sin x-2\ln x)\cdot\sqrt{x}}{2x(\sqrt{x}+1)^2}$$

(9) $f(x)=\dfrac{x^3+\cot x}{\ln x}$

$\frac{1}{x}$

$$f'(x) = \frac{(x^3+\cot x)'\ln x-(x^3+\cot x)(\ln x)'}{\ln^2 x} \qquad \frac{\cdot x}{\cdot x}$$

$$=\frac{(3x^2-\csc^2 x)x\ln x-(x^3+\cot x)}{x\cdot\ln^2 x}$$

(10). $f(x)=\dfrac{x\sin x+\cos x}{x\sin x-\cos x} = 1+\dfrac{2\cos x}{x\sin x-\cos x}$

$$f'(x)=\frac{2(\cos x)'(x\sin x-\cos x)-2\cos x(x\sin x-\cos x)'}{(x\sin x-\cos x)^2}$$

$-2x\cdot(\sin^2 x+\cos^2 x)-2\sin x\cos x$

$$=\frac{-2\sin x(x\sin x-\cos x)-2\cos x(\sin x+x\cos x+\sin x)}{(x\sin x-\cos x)^2}$$

$$=\frac{-2(x+\sin x\cos x)}{(x\sin x-\cos x)^2}$$

(11) $f'(x) = (e^x + \log_3 x)' \arcsin x + (e^x + \log_3 x)(\arcsin x)'$

$$= \left(e^x + \frac{1}{x\ln 3}\right)\arcsin x + (e^x + \log_3 x)\frac{1}{\sqrt{1-x^2}}$$

Notice : $(\arcsin x)' = \dfrac{1}{\sqrt{1-x^2}}$

$(\arctan x)' = \dfrac{1}{1+x^2}.$

(13) $f'(x) = \dfrac{(1+\tan x \sec x)(x-\csc x) - (x+\sec x)(1+\cot x \csc x)}{(x-\csc x)^2}$

(14) $f'(x) = \dfrac{(x+\sin x)' \arctan x - (x+\sin x)(\arctan x)'}{\arctan^2 x}$

$$= \frac{(1+x^2)(1+\cos x)\arctan x - (x+\sin x).}{(1+x^2)\arctan^2 x}$$

Find the derivatives of the following composite functions

(Chain Rule).

(1) $y=(2x^2-x+1)^2$;

(2) $y=e^{2x}\sin 3x$;

(3) $y=\dfrac{1}{\sqrt{1+x^3}}$;

(4) $y=\sqrt{\dfrac{\ln x}{x}}$;

(5) $y=\sin x^3$;

(6) $y=\cos\sqrt{x}$;

(7) $y=\sqrt{x+1}-\ln(x+\sqrt{x+1})$;

(8) $y=\arcsin(e^{-x^2})$;

(1) $y'=2\cdot(2x^2-x+1)\cdot(2x^2-x+1)'$

$=2(2x^2-x+1)(4x-1)$

(2) $y'=(e^{2x})'\sin 3x+e^{2x}(\sin 3x)'$

$=2e^{2x}\sin 3x+3\cos 3x\, e^{2x}$

$=(2\sin 3x+3\cos 3x)e^{2x}$

---

(3) $y=f(u(x))=\dfrac{1}{\sqrt{u(x)}}$ $\quad u(x)=1+x^3$

$\left(\dfrac{df}{dx}=\dfrac{df}{du}\cdot\dfrac{du}{dx}\right)$

$\dfrac{du}{dx}=u'=3x^2$. $\quad \dfrac{df}{du}=(u^{-\frac{1}{2}})'=-\frac{1}{2}u^{-\frac{3}{2}}$.

$y'=\dfrac{df}{dx}=\dfrac{df}{du}\cdot\dfrac{du}{dx}=-\frac{1}{2}(1+x^3)^{-\frac{3}{2}}\cdot 3x^2$

$=-\frac{3}{2}x^2(1+x^3)^{-\frac{3}{2}}$.

(4) $y=\sqrt{u(x)}$ $\quad u(x)=\dfrac{\ln x}{x}$

$y'=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$ $\quad u'(x)=\dfrac{1-x\ln x}{x^2}$

$=\dfrac{1}{2\sqrt{u}}\cdot u'$

$=\dfrac{1-\ln x}{2x^2}\cdot\sqrt{\dfrac{x}{\ln x}}$

---

(5) $y=f(u(x))=\sin u(x)$ $\quad u(x)=x^3$.

$y'=\dfrac{df}{du}\cdot\dfrac{du}{dx}=(\sin u)'\cdot u'=\cos x^3\cdot 3x^2$

$=3x^2\cdot\cos x^3$

(6) $y=\cos\sqrt{x}$ $\quad y=\cos u(x)$

$\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$ $\quad u(x)=\sqrt{x}$

$=-\sin u\cdot u'(x)=-\dfrac{\sin\sqrt{x}}{2\sqrt{x}}$

---

(7) $g(x)=\ln(x+\sqrt{x+1})$ $\quad u(x)=x+\sqrt{x+1}$

$g'(x)=\dfrac{dg}{du}\cdot\dfrac{du}{dx}=\dfrac{1+\frac{1}{2\sqrt{x+1}}}{x+\sqrt{x+1}}$ $\quad u'(x)=1+\frac{1}{2\sqrt{x+1}}$

$y'=\dfrac{1}{2\sqrt{x+1}}-\dfrac{1+2\sqrt{x+1}}{2\sqrt{1+x}(x+\sqrt{x+1})}$

(8) $y=\arcsin(e^{-x^2})$ $\quad u(x)=e^{-x^2}$, $u(x)=e^{v(x)}$, $v(x)=-x^2$

$=\arcsin(u(x))$ $\quad u'(x)=\dfrac{du}{dv}\cdot\dfrac{dv}{dx}=e^{v(x)}\cdot(-2x)=-2xe^{-x^2}$

$y'=\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$

$=\dfrac{1}{\sqrt{1-u^2}}\cdot(-2xe^{-x^2})$

$=\dfrac{-2xe^{-x^2}}{\sqrt{1-e^{-2x^2}}}\cdot\dfrac{e^{x^2}}{e^{x^2}}=\dfrac{-2x}{\sqrt{e^{2x^2}-1}}$

Equivalent (tool for limits).

$\lim\limits_{x\to 0} \frac{\sin x}{x} = 1.$ $(x\to 0)$. $x \sim \sin x.$

$x \sim \tan x.$

$\lim\limits_{x\to\infty} (1+\frac{1}{x})^x = e.$ $t = \frac{1}{x}.$

$\lim\limits_{t\to 0} (1+t)^{\frac{1}{t}} = e.$

$\Rightarrow \lim\limits_{\cancel{t}\, x\to 0} \frac{\ln(1+\cancel{t}\, x)}{\cancel{t}\, x} = 1.$ $\Rightarrow (x\to 0)$ $x \sim \ln(1+x).$

$x \sim e^x - 1$

$\alpha x \sim (1+x)^\alpha - 1.$

$(x\to 0)$: $x \sim \sin x \sim \tan x \sim \ln(1+x) \sim e^x - 1$

$\sim \frac{(1+x)^\alpha - 1}{\alpha}.$

$x \to 0$

$\ln(1+x^2) \sim x^2$

$e^{2x} - 1 \sim 2x$

$\tan x \sim x.$

e.g. $\lim\limits_{x\to 0} \frac{\ln(1+x^2)}{(e^{2x}-1)\tan x}$

$= \lim\limits_{x\to 0} \frac{x^2}{2x\cdot x} = \frac{1}{2}.$

EXAMPLE

$\lim\limits_{x\to 0} \frac{\sqrt{1+x} - e^{\frac{x}{3}}}{\ln(1+2x)}$ (easy, just do it.)

Exercise

Define $y = \underbrace{(1+x^2)\sin(2x^2+e^{x^2})}_{\mathrm{I}} + \underbrace{e^{\sin\ln x}}_{\mathrm{II}}$. Show that

$$\frac{dy}{dx} = 2x\sin(2x^2+e^{x^2}) + (1+x^2)\cos(2x^2+e^{x^2})(4x+2xe^{x^2}) + \frac{1}{x}e^{\sin\ln x}\cos\ln x.$$

✓

$\mathrm{II} = e^{\sin\ln x}$

$u(x) = \ln x$

$v(x) = \sin(u(x))$

$$v'(x) = \frac{dv}{du}\cdot\frac{du}{dx} = \cos u \cdot \frac{1}{x} = \cos\ln x \cdot \frac{1}{x}.$$

$$\mathrm{II}' = v'(x)\cdot e^{\sin\ln x} = \frac{1}{x}\cdot\cos\ln x\cdot e^{\sin\ln x}$$

$$\mathrm{I} = (1+x^2)\sin(2x^2+e^{x^2})$$

$w(x) = 2x^2+e^{x^2}$

$w'(x) = 4x+2xe^{x^2}$

$$\begin{aligned}\mathrm{I}' &= (1+x^2)'\sin(2x^2+e^{x^2}) + (1+x^2)\cdot(\sin(2x^2+e^{x^2}))' \\ &= 2x\sin(2x^2+e^{x^2}) + (1+x^2)\cdot(\sin w)'\cdot\frac{dw}{dx} \\ &= 2x\sin(2x^2+e^{x^2}) + (1+x^2)\cdot\cos(2x^2+e^{x^2})\cdot(4x+2x\cdot e^{x^2})\end{aligned}$$

$$y' = \mathrm{I}'+\mathrm{II}' = 2x\sin(2x^2+e^{x^2}) + (1+x^2)\cos(2x^2+e^{x^2})\cdot(4x+2xe^{x^2}) + \frac{1}{x}\cdot\cos\ln x\cdot e^{\sin\ln x}$$

#.

$f^{-1} = \sin^{-1} x \qquad f = \sin x$

Find the derivative of $\sin^{-1} x$. $(\arcsin x)$ $\qquad f' = \cos x$

$$(\arcsin x)' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

$$\cos^2 y + \sin^2 y = 1$$

Find out y'.

examples $y = y(x)$

(1) $y = x + \arctan y$;

(3) $\sqrt{x - \cos y} = \sin y - x$;

(5) $e^{x^2+y} - xy^2 = 0$;

(7) $2y\sin x + x\ln y = 0$;

exercises

(2) $y + xe^y = 1$;

(4) $xy - \ln(y+1) = 0$;

(6) $\tan(x+y) - xy = 0$;

(8) $x^3 + y^3 - 3axy = 0$.

reveal answers this weekend !!
(not compulsory)

(1). $\dfrac{d\,y(x)}{dx} = \dfrac{dx}{dx} + \dfrac{d\arctan y(x)}{dx}$

$y' = 1 + \dfrac{d\arctan y}{dy} \cdot \dfrac{dy}{dx}$

$y' = 1 + \dfrac{1}{1+y^2} \cdot y'$

$\left(1 - \dfrac{1}{1+y^2}\right) \cdot y' = 1$

$y' = \dfrac{1+y^2}{y^2}$

(3) $x - \cos y = (\sin y - x)^2$ $\qquad \dfrac{du^2}{dx}$

$\dfrac{d(x - \cos y)}{dx} = \dfrac{d(\sin y - x)^2}{dx}$ $\qquad u(x) = \sin y(x) - x$

$u' = \cos y \cdot y' - 1$

$1 - (-\sin y) \cdot y' = 2(\sin y - x) \cdot (\cos y \cdot y' - 1)$

$(-2(\sin y - x)\cos y + \sin y) \cdot y' = -2(\sin y - x) - 1$

$y' = \dfrac{2(\sin y - x) + 1}{2(\sin y - x)\cdot\cos y - \sin y}$

---

(5) $\dfrac{de^{x^2+y}}{dx} - \dfrac{dxy^2}{dx} = 0.$

$v(x) = x^2 + y(x)$

$v'(x) = 2x + 1 \cdot y'$

$y^2(x)' = 2y \cdot y'$

$v'(x) \cdot e^{x^2+y} - \left(\dfrac{dx}{dx} \cdot y^2 + \dfrac{dy^2(x)}{dx} \cdot x\right) = 0$

$(2x + y')e^{x^2+y} - (y^2 + 2xy \cdot y') = 0$

$(e^{x^2+y} - 2xy) \cdot y' = y^2 - 2xe^{x^2+y}$

$y' = \dfrac{y^2 - 2xe^{x^2+y}}{e^{x^2+y} - 2xy}$

(7) $2y\sin x + x\ln y = 0$

$2 \cdot \dfrac{dy\sin x}{dx} + \dfrac{dx\ln y}{dx} = 0$

$(\ln y(x))' = \dfrac{1}{y} \cdot y'$

$2 \cdot (y'\sin x + y\cos x) + \ln y + x(\ln y(x))' = 0$

$2\sin x \cdot y' + \dfrac{x}{y} \cdot y' = -\ln y - 2y\cos x$ $\qquad \dfrac{\cdot y}{\cdot y}$

$(2y\sin x + x) \cdot y' = -(y\ln y + 2y^2\cos x)$

$y' = -\dfrac{y\ln y + 2y^2\cos x}{2y\sin x + x}$

(2) $\frac{dy(x)}{dx} + \frac{d\,xe^{y(x)}}{dx} = 1$

$y' + e^y + xy'e^y = 0$

$y' = \frac{-e^y}{1+xe^y}$

(4) $\frac{d\,xy(x)}{dx} - \frac{d\ln(y(x)+1)}{dx} = 0$

$x - \frac{1}{y+1} = \frac{xy+x-1}{y+1}$

$y + xy' - \frac{y'}{y+1} = 0$

$y' = -y \cdot \frac{y+1}{xy+x-1}$

(6) $\frac{d\tan(x+y)}{dx} - \frac{dxy}{dx} = 0$

$\sec^2(x+y)\cdot(1+y') - y - xy' = 0$

$y' = \frac{\sec^2(x+y)-y}{x-\sec^2(x+y)}$

(8) $\frac{dx^3}{dx} + \frac{dy^3}{dx} - 3a\frac{dxy}{dx} = 0$

$3x^2 + 3y^2y' - 3a(y+xy') = 0$

$(y^2-ax)\,y' = ay - x^2$

$y' = \frac{ay-x^2}{y^2-ax}$

$u = xy^3$

$u' = (x)'\cdot y^3 + x\cdot(y^3)'$

$= y^3 + 3x\cdot y^2\cdot y'$

Review implicit derivative
Find out dy/dx by implicit differentiation:

$$2x+5 = \cos(xy^3)$$

$$\frac{d(2x+5)}{dx} = \frac{d\cos(x\cdot y^3(x))}{dx}$$

$$2 = \frac{d\cos(u(x,y(x)))}{dx}$$

$$2 = -\sin u \cdot u'$$

$$2 = -\sin xy^3 \cdot (y^3 + 3x \cdot y^2 \cdot y')$$

$$y' = \frac{2 + \sin xy^3 \cdot y^3}{-3xy^2 \cdot \sin xy^3}$$

Theorems review

Theorem (Fermat's Theorem)

If f has a local maximum of minimum at an interior point c, and if $f'(c)$ exists, then $f'(c) = 0$.

As a result, we obtain a basic approach to find the global maximum and minimum of a differentiable function f on a closed interval $[a, b]$ is:

1. Find all critical points of f in (a, b), and the respective function values.
2. Find the function values of f at the boundary points of the interval $[a, b]$.
3. Just compare these function values above to find the largest (global maximum) and smallest (global minimum).

Rolle's Theorem

Combining the extreme value theorem and Fermat's theorem, it is easy to conclude Rolle's theorem.

Theorem (Extreme Value Theorem)

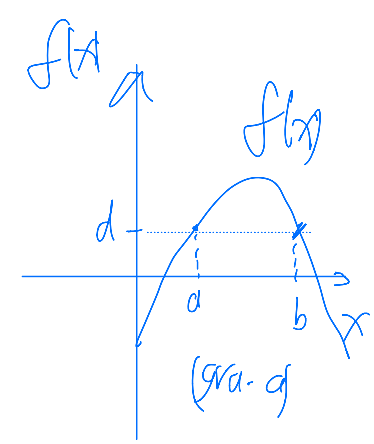
If f is continuous on a closed interval, then f attains an global maximum $f(c)$ and an global minimum $f(d)$ at some numbers c and d in $[a, b]$.

Theorem (Fermat's Theorem)

If f has a local maximum or local minimum at an interior point c, and if $f'(c)$ exists, then $f'(c) = 0$.

Theorem (Rolle's Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and $f(a) = f(b)$ and $a < b$, then $f'(c) = 0$ for some number $c \in (a, b)$.



$f(a) = f(b)$ $\exists c$
$c \in (a, b)$
$f'(c) = 0.$

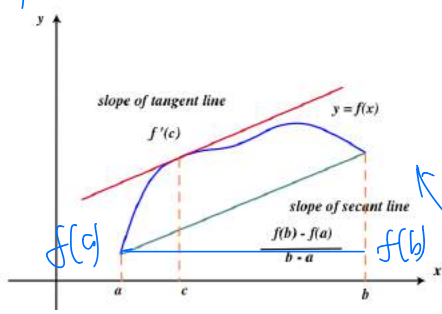
Lagrange M. V. Thm.

Theorem (Mean Value Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some $\boxed{c \in (a, b)}$, or equivalently $f(b) - f(a) = f'(c)(b - a)$.



$f(a) = f(b) = d$

$y = d$

yra b.

Cauchy M. V. Thm.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \quad \text{where } c \in (a, b).$$

(Cauchy)

$g(x) = x \qquad g'(x) = 1 \longrightarrow$ (Lagrange).

Utilities of derivatives

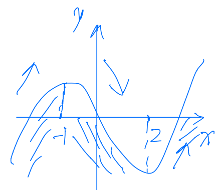
Find the intervals of increasing/decreasing and local extrema of the following functions

(1) $y = 2x^3 - 3x^2 - 12x + 1$; (2) $y = x + \sin x$;

(3) $y = \sqrt{x} \ln x$; (4) $y = x^n e^{-x}$ $(n \in \mathrm{N}^+)$;

(1) $y' = 6x^2 - 6x - 12$
$= 6(x^2 - x - 2)$
$= 6(x+1)(x-2)$

$\begin{matrix} 1 & 1 \\ 1 & -2 \end{matrix}$



points $y' = 0$: $x = -1$, $x = 2$.

$x \in (-\infty, -1) \cup (2, +\infty)$ $y' > 0$, increasing

$x \in [-1, 2]$ $y' \leq 0$, decreasing

local $y_{max} = y|_{x=-1}$ local $y_{min} = y|_{x=2}$.

(4) $y' = n \cdot x^{n-1} \cdot e^{-x} + (-1) \cdot x^n e^{-x}$
$= (n - x) \cdot x^{n-1} \cdot e^{-x}$

$n-1 \begin{cases} \text{odd} & ? \\ \text{even.} & ? \end{cases}$

points $y' = 0$: $x = n$ & $x = 0$

n is even: $x \in (-\infty, 0] \cup [n, +\infty)$ $y' \leq 0$ ↓
$x \in (0, n)$ $y' > 0$ ↑

local $y_{min} = y|_{x=0}$, local $y_{max} = y|_{x=n}$

n is odd: $x \in (-\infty, n]$ $y' \geq 0$ ↗
$x \in (n, +\infty)$ $y' < 0$ ↘

local $y_{max} = y|_{x=n}$

Utilities of mean value theorems

$f(b)-f(a)=f'(c)(b-a) \quad c\in(a,b)$

Prove the following inequality by using (Lagrange) mean value theorem

(1) $|\sin x-\sin y|\le|x-y|$;

(2) $ny^{n-1}(x-y)<x^n-y^n<nx^{n-1}(x-y) \quad (n>1,\ x>y>0)$;

(3) $\dfrac{b-a}{b}<\ln\dfrac{b}{a}<\dfrac{b-a}{a} \quad (b>a>0)$;

$\ln\frac{b}{a}=\ln b-\ln a$

$(\ln c)'=\frac{1}{c}$.

(4) $e^x>1+x \quad (x>0)$.

e.g. (1) $f(t)=\sin t \quad f'(t)=\cos t. \quad z\in(x,y)$

$|f(x)-f(y)|=|\cos z\cdot(x-y)|\le|x-y|$

for $\forall z\in\mathbb{R}.\ |\cos z|\le 1$

(2) $f(t)=t^n \quad f'(t)=n\cdot t^{n-1}$

$f(x)-f(y)=f'(z)(x-y)=nz^{n-1}(x-y)$ $\quad D=\{n>1,\ x>z>y>0\}$

$g(z)=nz^{n-1}\underbrace{(x-y)}_{\text{constant}}$

$g'(z)=\underset{>0}{n}\cdot\underset{>0}{(n-1)}\cdot\underset{>0}{z^{n-2}}\cdot\underset{>0}{(x-y)}>0 \quad g(z)\uparrow$ in the D.

$ny^{n-1}(x-y)<nz^{n-1}(x-y)<nx^{n-1}(x-y)$

(4) $1=e^0$

$e^x-e^0>x-0$ $\quad c\in(0,x)$

$f(x)=e^x,\quad f'(x)=e^x$ $\quad x>c>0,$

$e^c>1.$

$f(x)-f(0)=f'(c)(x-0)$

$=e^c\cdot x>x.$

A little bit preview

Anti derivative integration

Example 1. $f(x) = 3x^2 \longleftrightarrow$ Solve the antiderivative problem: $\frac{d}{dx}(\overset{g(x)}{?}) = 3x^2$

Example 2.

(i) $g(x) = 2\cos x \longleftrightarrow$ Solve the antiderivative problem: $\frac{d}{dx}(?) = 2\cos x$

(ii) $h(x) = x + e^{2x} \longleftrightarrow$ Solve the antiderivative problem: $\frac{d}{dx}(\overset{f(x)}{?}) = x + e^{2x}$

$3 = 2+1$

$\frac{d\,g(x)}{dx} = 3x^2$

$(x^n)' = n \cdot x^{n-1}$

inverse calculation

power +1.

$g(x) = \frac{3}{2+1}x^3 = x^3 \boxed{+C}$ (Constant)

$(x^3+3)' = 3x^2$

$(x^3+10^6)' = 3x^2$

(ii) $\frac{df(x)}{dx} = x + e^{2x}$

$(e^{2x})' = 2 \cdot e^{2x}$

$\frac{d?_1}{dx} = x$

$\frac{d?_2}{dx} = e^{2x}$

$(\frac{1}{2}e^{2x} + C_2)' = \frac{2}{2}e^{2x} + 0$

$= e^{2x}$

$?_2 = \frac{1}{2}e^{2x} + C_2$

$?_1 = \frac{1}{1+1}x^2 + C_1 = \frac{1}{2}x^2 + C_1$

$C_1 + C_2 = C$

(Const)

$? = \frac{1}{2}(x^2 + e^{2x}) + C_1 + C_2$

$= \frac{1}{2}(x^2 + e^{2x}) + C.$

Theorem (1st/2nd order condition)

Suppose function f is twice differentiable over an open interval I. Then, the following statements are equivalent:

(a) f is convex.

(b) $f(y) \geq f(x) + f'(x)(y-x)$, for all x and y in I.

(c) $f''(x) \geq 0$, for all x in I.

Lagrangen M.V Thm $\exists\ \xi \in (a,b)$ s.t.

$$f(b) - f(a) = f'(\xi)(b-a)$$

Exercise

If f is twice differentiable on (a,b) and continuous on $[a,b]$, then

$$f(b) - f(a) - f'(a)(b-a) = \frac{f''(c)}{2}(b-a)^2 \quad \text{eqn.(1)}$$

for some $c \in (a,b)$.

Hint: Consider

$$h(x) = f(x) - f(a) - f'(a)(x-a) - \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}(x-a)^2$$

$$h'(x) = f'(x) - f'(a) - 2 \cdot \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}(x-a)$$

What if you now apply the Rolle's Theorem? What is $h''(x)$?

Lagrange M.V. Thm

2nd

Rollés Thm.

$f(a) = f(b)$

$\exists\ \eta \in (a,b)$

s.t. $f'(\eta) = 0$

$$h(b) = f(b) - f(a) - f'(a)(b-a) - \frac{f(b)-f(a)-f'(a)(b-a)}{\cancel{(b-a)^2}} \cancel{(b-a)^2} = 0.$$

$$h(a) = f(a) - f(a) - f'(a)(a-a) - \cdots\cdots (a-a) = 0 \quad = h(b)$$

($\exists\ d \in (a,b)$, s.t.) Rolle's Thm

$$h'(d) = f'(d) - f'(a) - 2 \cdot \frac{f(b)-f(a)-f'(a)(b-a)}{(b-a)^2} \cdot (d-a) = 0$$

$$\Rightarrow \quad \frac{f'(d) - f'(a)}{d-a} = 2 \cdot \frac{f(b)-f(a)-f'(a)(b-a)}{(b-a)^2}$$

Lagrange M.V. Thm.

($\exists$ $c \in (a,d)$, s.t.)

$$\frac{f'(d)-f'(a)}{d-a} = f''(c) = 2\cdot\frac{f(b)-f(a)-f'(a)(b-a)}{(b-a)^2}$$

$$\Rightarrow f(b)-f(a)-f'(a)(b-a) = \frac{f''(c)}{2}(b-a)^2 \quad \#.$$

For any x and y in I, we have

$$f(y) - f(x) - f'(x)(y-x) = \frac{f''(z)}{2}(y-x)^2$$

for some z in $[x, y]$ (which means z is in I and $f''(z) \geq 0$). Then

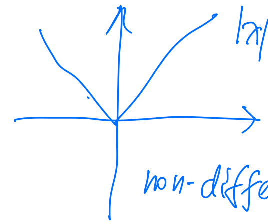
$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(z)}{2}(y-x)^2 \geq f(x) + f'(x)(y-x).$$

Theorem (also holds for non-differentiable function)

Suppose function f is convex on interval I. If x^* is a local minimum over I, then x^* is also a global minimum of f over I.

Theorem

If function f is convex and differentiable over an interval I. Then any point x^* that satisfies $f'(x^*) = 0$ holds that $f(x^*)$ is a global minimum.



non-differentiable $x=0$.

Exercise

Provide examples of $f(x)$ for the following cases respectively

(1) • $f(x)$ is convex, but not differentiable $|x|$

(2) • $f(x)$ is convex and differentiable, but $f'(x)$ is non-differentiable $x^{\frac{4}{3}}$

(3) • $f(x)$ is convex and differentiable, and $f^{(n)}(x)$ is differentiable for all positive integer n. e^x

(2) non-diff ...

$f(x)=x^n, f'(x)=nx^{n-1}$ $\quad x^{n-1}$ $\quad f'(x)=x^{\frac{1}{3}}$ $\quad$ n-d $x=0$ $\quad f(x)=\frac{4}{3}x^{\frac{4}{3}}$

$f(x)=\frac{4}{3}x^{\frac{4}{3}}+C$

(3) $f(x)=e^{nx}$ $\quad f'(x)=ne^{nx}.\ f^{(n)}=n^n e^{nx}$ ✓.

Theorem (Macho L'Hôpital's Rule, one-side)

Suppose that f and g are continuous on a closed interval $[a,b]$, and are differentiable on the open interval (a,b). Suppose that $g'(x)$ is never zero on (a,b) and $\lim_{x\to a^+}\frac{f'(x)}{g'(x)}$ exists, and that $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$. Then

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}.$$

This theorem doesn't require anything about $g'(a)$, just about how g' behaves to the right of a.

The conclusion relates limit of $f(x)/g(x)$ to another one-side limit of $f(x)'/g(x)'$, and not to the value of $f'(a)/g'(a)$.

Exercise/Tutorial

Prove Macho L'Hôpital's rule.

Hint: Although we do not suppose what is $f(a)$ and $g(a)$, we can define

$$F(x) = \begin{cases} 0 & x = a \\ f(x) & x > a \end{cases} \quad \text{and} \quad G(x) = \begin{cases} 0 & x = a \\ g(x) & x > a \end{cases}.$$

Then try to prove the following theorem and apply it.

Theorem (cauchy's mean value theorem)

If $F(x)$ and $G(x)$ are continuous on $[a,b]$ and differentiable on (a,b), then there is a point c in (a,b) such that

$$(F(b)-F(a))G'(c) = (G(b)-G(a))F'(c).$$

(when $G(x)=x$, this is the same as the usual mean value theorem)

$c \in (a, x)$.

$$\frac{F(x)-F(a)}{G(x)-G(a)} = \frac{F'(c)}{G'(c)}$$

Baby L'Hôpital's Rule

Proof.

Since $f(a) = g(a) = 0$, we have

$$\lim_{x\to a}\frac{f(a)}{g(a)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a}\frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

$$= \frac{\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}{\lim_{x\to a}\frac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

where the last step use the continuity of f' and g'. □

$(F(a) = G(a) = 0)$

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{F(x)}{G(x)} = \lim_{x\to a^+}\frac{F(x)-F(a)}{G(x)-G(a)} = \lim_{x\to a^+}\frac{F'(c)}{G'(c)}$$

$x \to a^+$. $c \in (a, x) \Rightarrow c \to a^+$

$$= \lim_{c\to a^+}\frac{F'(c)}{G'(c)} = \lim_{c\to a^+}\frac{f'(c)}{g'(c)}$$

$c \to x$

$$= \lim_{x\to a^+}\frac{f'(x)}{g'(x)}$$

#.

Transforming functions for L'Hospital $\left(\frac{0}{0}\ \cdot\ \frac{\infty}{\infty}\right)$

$0\cdot\infty$; $\infty-\infty$; $\infty^0,\ 1^\infty,\ 0^0$

Examples

(1) $\lim_{x\to0^+} x\ln x$ $\quad x\to0^+$ $\quad 0\cdot\infty$

$x=\frac{1}{\frac{1}{x}}$ $\quad \frac{1}{\infty}$ $\quad \frac{\infty}{\infty}$

$=\lim_{x\to0^+}\frac{\ln x}{\frac{1}{x}}$

$\overset{L'H}{=}\lim_{x\to0^+}\frac{\frac{1}{x}}{\frac{1}{-x^2}}$

$=\lim_{x\to0^+}(-x)=0.$

(2) $\lim_{x\to0^+}\left(\cot x-\frac{1}{x}\right)$

$x\to0^+$ $\quad \frac{\cos x}{\sin x}$ $\quad (\infty-\infty)$ $\quad \frac{1}{0}-\frac{1}{0}$ $\quad \frac{0-0}{0}\to\left(\frac{0}{0}\right)$

$=\lim_{x\to0^+}\left(\frac{\cos x}{\sin x}-\frac{1}{x}\right)$

$=\lim_{x\to0^+}\left(\frac{x\cos x-\sin x}{x\sin x}\right)$

$x\cos x-\sin x\to0\ (x\to0)$

$x\sin x\to0\ (x\to0)$

$\left(\frac{0}{0}\right)$

$\overset{L'H}{=}\lim_{x\to0^+}\frac{\cancel{\cos x}-x\sin x-\cancel{\cos x}}{\sin x+x\cos x}$

$=\lim_{x\to0^+}\frac{-x}{1+\frac{x}{\sin x}\cos x}=0.$

---

(3) $\lim_{x\to0^+}x^x$

$\lim f(x)^{g(x)}$

$=\lim e^{\ln f(x)^{g(x)}}$

$=\lim e^{g(x)\cdot\ln f(x)}$

$=e^{\lim g(x)(\ln f(x))}$

$=e^{\lim_{x\to0^+}x\ln x}$

$\left(\lim_{x\to0^+}x\ln x=0\right)$

$=e^0=1$

(4) $\lim_{x\to\frac{\pi}{2}^+}(\sin x)^{\tan x}$

$=e^{\lim_{x\to\frac{\pi}{2}^+}\tan x\cdot\ln\sin x}$ $\quad\to\quad \frac{\ln\sin x}{\cot x}$ $\quad\left(\frac{\infty}{\infty}\right)$

$\lim_{x\to\frac{\pi}{2}^+}\frac{\ln\sin x}{\cot x}$

$\overset{L'H}{=}\lim_{x\to\frac{\pi}{2}^+}\frac{(\ln\sin x)'}{(\cot x)'}$ (skip steps)

$=0.$

$\lim_{x\to\frac{\pi}{2}^+}(\sin x)^{\tan x}=e^0=1$

Exercises for L'Hospital rule.

(1) $\lim_{x \to 0} x \cot 2x$;

(2) $\lim_{x \to \pi} (\pi - x) \tan \frac{x}{2}$;

(3) $\lim_{x \to 0+} \left(\frac{1}{x}\right)^{\tan x}$;

(4) $\lim_{x \to 0+} \left(\ln \frac{1}{x}\right)^{\sin x}$;

(1) $\lim_{x \to 0} \frac{x}{\sin 2x} \cdot \lim_{x \to 0} \cos 2x$ $\overset{1}{\quad}$

$= \lim_{x \to 0} \frac{x}{\sin 2x} \quad \left(\frac{0}{0}\right)$

$\overset{L'H}{=} \lim_{x \to 0} \frac{1}{2\cos 2x} = \frac{1}{2}$

(2) $\lim_{x \to \pi} \frac{(\pi - x)}{\cos \frac{x}{2}} \cdot \lim_{x \to \pi} \sin \frac{x}{2}$ $\overset{1}{\quad}$

$= \lim_{x \to \pi} \frac{\pi - x}{\cos \frac{x}{2}} \quad \left(\frac{0}{0}\right)$

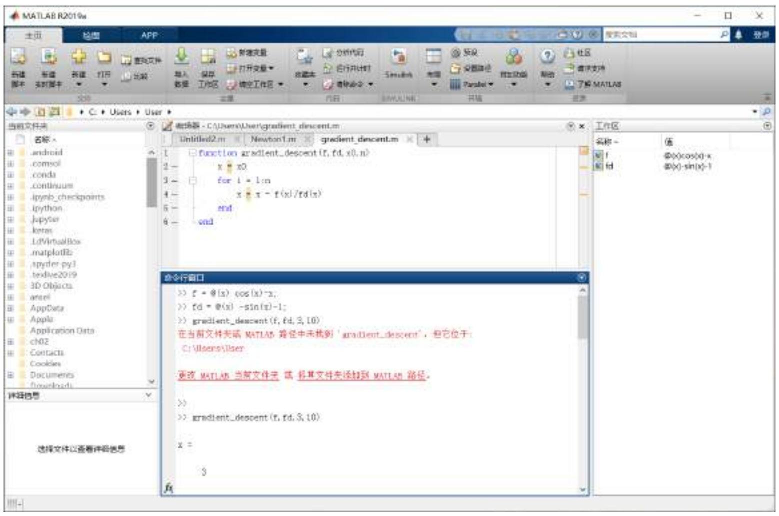
$\overset{L'H}{=} \lim_{x \to \pi} \frac{-1}{-\frac{1}{2} \sin \frac{x}{2}} = 2.$

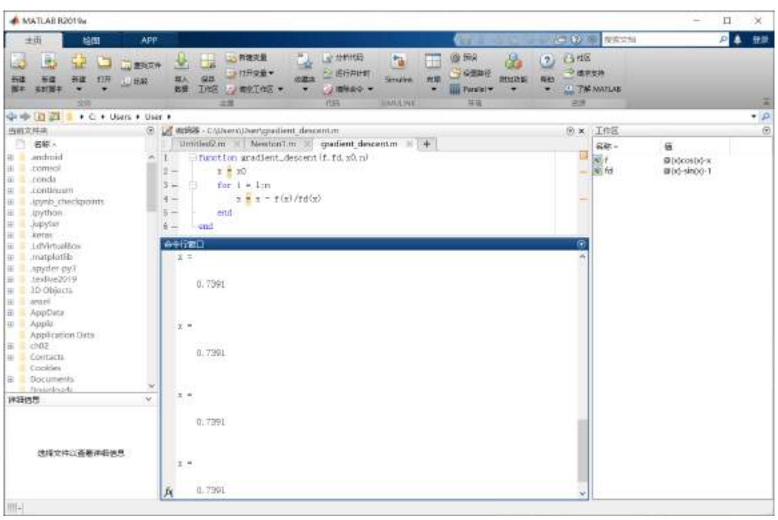
Exercise

Find an approximate value of the root of the equation

$$\cos x - x = 0$$

by the Newton's Method (using calculator or MATLAB).





Newton's method may fail even if $f'(x_k) \neq 0$ for any k when x_0 is far way from the solution.

Exercise

Consider solving

$$f(x) = x^{\frac{1}{3}} = 0$$

by Newton's method with initial point $x_0 = 1$.

Exercise

Compare the convergence of Newton's method with bisection method (Lecture 08). Which one is faster?

1st-order deriv $\qquad\qquad$ indefinite integral

$$\frac{d}{dx}\frac{1}{p+1}x^{p+1} = x^p \quad \overset{p\neq -1}{\Longleftrightarrow} \quad \int x^p dx = \frac{1}{p+1}x^{p+1} \boxed{+ C}$$

$$\frac{d}{dx}e^x = e^x \quad \Longleftrightarrow \quad \int e^x dx = e^x + C$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \quad \Longleftrightarrow \quad \int \frac{1}{x}dx = \ln|x| + C$$

$$\frac{d}{dx}\sin x = \cos x \quad \Longleftrightarrow \quad \int \cos x dx = \sin x + C$$

$$\frac{d}{dx}[-\cos x] = \sin x \quad \Longleftrightarrow \quad \int \sin x dx = -\cos x + C$$

$$\frac{d}{dx}\tan x = \sec^2 x \quad \Longleftrightarrow \quad \int \sec^2 x dx = \tan x + C$$

$$\frac{d}{dx}\cot x = -\csc^2 x \quad \Longleftrightarrow \quad \int \csc^2 x dx = -\cot x + C$$

$$\frac{d}{dx}\sec x = \sec x \cdot \tan x \quad \Longleftrightarrow \quad \int \sec x \tan x dx = \sec x + C$$

$$\frac{d}{dx}\csc x = -\cot x \csc x \quad \Longleftrightarrow \quad \int \cot x \csc x dx = -\csc x + C$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \quad \Longleftrightarrow \quad \int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2} \quad \Longleftrightarrow \quad \int \frac{dx}{1+x^2} = \arctan x + C$$

Exercise

Check the formula

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

$x \neq 0$ $(-\infty, 0)$ $(0, +\infty)$

and

$$\int \frac{1}{x}dx = \ln|x| + C.$$

$x > 0$, $(\ln x)' = \frac{1}{x}$ ✓ $\Leftrightarrow \int \frac{dx}{x} = \ln x + C$ $(x>0)$

$x<0$, $|x| = -x$, $[\ln(-x)]' = (-1)\cdot\frac{1}{-x} = \frac{1}{x}$

$\Leftrightarrow \int \frac{dx}{x} = \ln(-x) + C$ $(x<0)$

$\Leftrightarrow \int \frac{dx}{x} = \ln|x| + C$ ✓ $\frac{d}{dx}\ln|x| = \frac{1}{x}$ ✓

Exercise

Find the following indefinite integral

1. $\int \frac{x}{\sqrt{x^2+1}}dx$
2. $\int \sec x dx$ (hint: let $u = \sec x + \tan x$)
3. $\int \sin^3\theta d\theta$ $\sin^3\theta = \sin^2\theta \cdot \sin\theta = (1-\cos^2\theta)\cdot\sin\theta$
   $d\cos\theta = -\sin\theta d\theta$
   $d\theta = \frac{1}{-\sin\theta}d\cos\theta$
4. $\int \cos^3\theta d\theta$
5. $\int xe^{6x^2}dx$

① $u = \sqrt{x^2+1} = (x^2+1)^{\frac{1}{2}}$

$du = 2x\cdot(x^2+1)^{-\frac{1}{2}}\cdot(\frac{1}{2})dx$

$= \frac{x}{\sqrt{x^2+1}}dx$

$dx = \frac{\sqrt{x^2+1}}{x}du$

$\int \frac{x}{\sqrt{x^2+1}}dx$

$= \int \frac{x}{u}\cdot\frac{u}{x}du$

② $du = \left[\frac{d}{dx}\sec x + \frac{d}{dx}\tan x\right]dx$

$du = (\sec x\tan x + \sec^2 x)dx$

$dx = \frac{du}{\sec x\underbrace{(\sec x + \tan x)}_{=u}} = \frac{du}{\sec x\cdot u}$

$\int \sec x dx = \int \frac{\sec x\, du}{\sec x\cdot u}$

$= \int 1 \cdot du$

$= u + C = \sqrt{x^2+1} + C$

$= \int \frac{du}{u}$

$= \ln|\sec x + \tan x| + C$

③ $u = \cos\theta$

$\int \sin^3\theta \, d\theta = \int \frac{(1-\cos^2\theta)\cdot\sin\theta}{-\sin\theta} d\cos\theta$

$= \int (\cos^2\theta - 1) \, d\cos\theta$

$= \int u^2 - 1 \, du$

$= \frac{1}{3}u^3 - u + C$

$= \frac{\cos^3\theta}{3} - \cos\theta + C$

⑤ $\frac{d}{dx} 6x^2 = 12x \, dx \qquad u = 6x^2$

$dx = \frac{1}{12x} d6x^2$

$\int x e^{6x^2} dx$

$= \frac{1}{12}\int \frac{x}{x} e^u du$

$= \frac{1}{12} e^u + C$

$= \frac{1}{12} e^{6x^2} + C$

Exercise

Find the following indefinite integral

1. $\displaystyle\int x\sqrt{x+1}dx$
2. $\displaystyle\int \ln x dx$

① $u = x+1 \qquad du = dx$

$\int (u-1)\cdot\sqrt{u} \, du$

$= \int u\cdot\sqrt{u} - \sqrt{u} \, du$

$= \int u^{\frac{3}{2}} - u^{\frac{1}{2}} du$

$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + C = \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + C$

② $\int 1 \cdot \ln x \, dx$

$f(x) = \ln x \qquad g(x) = x$

$f'(x) = \frac{1}{x} \qquad g'(x) = 1$

$= \int f \cdot g' \, dx$

$= f \cdot g - \int f' \cdot g \, dx$

$= x \ln x - \int \frac{x}{x} dx$

$= x \ln x - x + C$

Theorem (Fundamental Theorem of Calculus)

Let f be a continuous function on the closed interval $[a, b]$. If $F(x)$ is an antiderivative of f, i.e., $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a),$$

which is often denoted as $F(x)\big|_a^b$ or $\big[F(x)\big]_a^b$.

In other words, whenever you can find

$$\int f(x)dx = F(x) + C,$$

it is just one step further to find the corresponding definite integral:

$$\int_a^b f(x)dx = F(b) - F(a).$$

$$\frac{d}{dx}\frac{1}{p+1}x^{p+1} = x^p \quad \overset{p\neq-1}{\iff} \quad \int x^p dx = \frac{1}{p+1}x^{p+1} + C$$

$$\frac{d}{dx}e^x = e^x \quad \iff \quad \int e^x dx = e^x + C$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \quad \iff \quad \int \frac{1}{x}dx = \ln|x| + C$$

$$\frac{d}{dx}\sin x = \cos x \quad \iff \quad \int \cos x dx = \sin x + C$$

$$\frac{d}{dx}[-\cos x] = \sin x \quad \iff \quad \int \sin x dx = -\cos x + C$$

$$\frac{d}{dx}\tan x = \sec^2 x \quad \iff \quad \int \sec^2 x dx = \tan x + C$$

$$\vdots$$

$$\frac{d}{dx}a^x = a^x \ln a.$$

Find the derivative of F(x)

a is Const.

$$F(x)=\int_a^{\ln x} f(t)dt\,;$$

$$F(x)=\int_a^{\left(\int_0^x \sin^2 t dt\right)} \frac{1}{1+t^2}dt\,.$$

$\sin^2 x$

(1) $F'(x)=f(\ln x)\cdot(\ln x)'-f(a)\cdot a'$

$=\frac{1}{x}f(\ln x).$

(2) $\frac{1}{1+\int_0^x \sin^2 t dt}\cdot \sin^2 x$

$=\frac{4\sin^2 x}{4+(x-\sin x\cos x)^2}$

Find the following limits.

(1) $\lim\limits_{n\to\infty}\left(\frac{1}{n^2}+\frac{2}{n^2}+\frac{3}{n^2}+\cdots+\frac{n-1}{n^2}\right);$ $\frac{1}{n}$

(2) $\lim\limits_{n\to\infty}\frac{1^p+2^p+3^p+\cdots+n^p}{n^{p+1}}\quad(p>0);$

(1) $\lim\limits_{n\to\infty}\left(\frac{0}{n}+\frac{1}{n}+\cdots\frac{n-1}{n}\right)\cdot\frac{1}{n}$

$=\lim\limits_{n\to\infty}\sum\limits_{i=0}^{n-1}\left(\frac{i}{n}\right)\cdot\frac{1}{n}=\int_0^1 x\,dx=\frac{1}{2}$

(2) $\lim\limits_{n\to\infty}\left(\frac{1^p}{n^p}+\frac{2^p}{n^p}+\cdots\frac{n^p}{n^p}\right)\cdot\frac{1}{n}$

$=\lim\limits_{n\to\infty}\sum\limits_{i=1}^{n}\left(\frac{i}{n}\right)^p\cdot\frac{1}{n}=\int_0^1 x^p dx=\frac{1}{p+1}$

Find the integral of the following functions.

(1) $\int_0^1 x^2(2-x^2)^2 dx$;

(2) $\int_1^2 \frac{(x-1)(x^2-x+1)}{2x^2} dx$;

(3) $\int_0^2 (2^x+3^x)^2 dx$;

(4) $\int_0^{\frac{1}{2}} x(1-4x^2)^{10} dx$;

(5) $\int_{-1}^1 \frac{(x+1)dx}{(x^2+2x+5)^2}$;

(6) $\int_0^1 \arcsin x dx$;

(7) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x}{\cos^2 x} dx$;

(8) $\int_0^{\frac{\pi}{4}} x \tan^2 x dx$;

(9) $\int_0^{\frac{\pi}{2}} e^x \sin^2 x \, dx$;

(10) $\int_1^e \sin(\ln x) \, dx$;

(11) $\int_0^1 x^2 \arctan x dx$;

(12) $\int_1^{e+1} x^2 \ln(x-1) dx$。

(13) $\int_0^{\sqrt{\ln 2}} x^3 e^{-x^2} dx$;

(14) $\int_0^1 e^{2\sqrt{x+1}} dx$;

(15) $\int_0^1 \frac{dx}{\sqrt{1+e^{2x}}}$;

(16) $\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{\sqrt{(1-x^2)^3}}$;

(2) $\int_1^2 \left(\frac{x}{2}-1+\frac{1}{x}-\frac{1}{2x^2}\right) dx = \ln 2 - \frac{1}{2}$

(4) $u = 1-4x^2 \Rightarrow du = -8x\,dx \Rightarrow dx = -\frac{du}{8x}$

$= \int_0^{\frac{1}{2}} \left(-\frac{1}{8}\right)\cancel{\frac{x}{x}} \cdot u^{10} du = -\frac{1}{88}(1-4x^2)^{11}\Big|_0^{\frac{1}{2}} = \frac{1}{88}$

(5) $x+1$. $u = (x+1)^2$ $\quad dx = \frac{du}{2(x+1)}$

$x^2+2x+5 = (x+1)^2+4$

$= \frac{1}{2}\int_{-1}^{1} \frac{du}{(u^2+4)^2} = -\frac{1}{2(x^2+2x+5)}\Big|_{-1}^{1} = \frac{1}{16}$

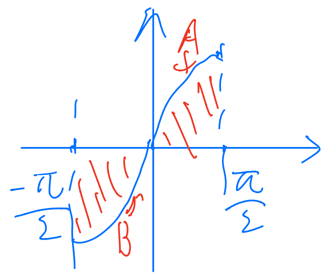
Odd func.

(7) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x}{\cos^2 x}\, dx = 0$

$f(x) = \sin x \quad x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Sym. Interval

$g(-x) = -g(x)$



(16) $I = \int_1^e \sin(\ln x)\, dx$

$f(x) = x \quad g(x) = \sin(\ln x)$

$f'(x) = 1 \quad g'(x) = \frac{1}{x}\cos(\ln x)$

$I = x\sin(\ln x)\Big|_1^e - \int_1^e 1\cdot\cos(\ln x)\, dx$

$\int_1^e \cos(\ln x)\, dx$

$h(x) = \cos(\ln x)$

$h'(x) = \frac{-1}{x}\sin(\ln x)$

$= x\cos(\ln x)\Big|_1^e + \overset{= I}{\int_1^e 1\cdot\sin(\ln x)\, dx}$

$I = x\sin(\ln x)\Big|_1^e - x\cos(\ln x)\Big|_1^e - I$

$2I = - \quad - \quad - \quad -$

$I = \frac{e(\sin 1 - \cos 1) + 1}{2}$

(13) $\int_0^{\sqrt{\ln 2}} x^3 e^{-x^2} dx$ $\qquad$ $dx^2 = 2x\,dx \Rightarrow dx = \frac{dx^2}{2x}$

$$= -\frac{1}{2} x^2 e^{-x^2} \Big|_0^{\sqrt{\ln 2}} + \frac{1}{2} \int_0^{\sqrt{\ln 2}} e^{-x^2} dx^2$$

$$= \frac{-\ln 2}{4} - \frac{1}{2} e^{-x^2} \Big|_0^{\sqrt{\ln 2}} = \frac{1 - \ln 2}{4}$$

Find the counter example to the following propositions:

differentiable, continuous, (Riemann integrable)

1. f(x) is differentiable at x=a, then f'(x) is continuous at x=a;
2. The derivative of f(x) f'(x) exists, ~~then~~ f'(x) is Riemann integrable.

1. $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & , x \neq 0 \\ 0 & , x = 0 \end{cases}$ $\quad f'(x) \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0 & , x = 0 \end{cases}$

$x \to 0^-$ or 0^+ $\quad \lim\limits_{x \to 0^- or 0^+} f'(x) = \lim\limits_{x \to 0^- or 0^+} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$ does not exist.

$f'(x)$ is not C... at $x=0$.

2. $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ $\quad f'(x) = \begin{cases} 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x} \cos\left(\frac{1}{x^2}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$

$n \in \mathbb{Z}^+ \quad f'\left(\frac{1}{\sqrt{n\pi}}\right) = \frac{2}{\sqrt{n\pi}} \underbrace{\sin(n\pi)}_{=0} - 2\sqrt{n\pi} \underbrace{\cos(n\pi)}_{=(-1)^n}$

$= (-1)^{n+1} \, 2\sqrt{\pi} \cdot \sqrt{n}.$

$n \to \infty \quad \left| f'\left(\frac{1}{\sqrt{n\pi}}\right) \right| = 2\sqrt{\pi} \cdot \sqrt{n} \to \infty$

$f'(x)$ is not Riemann integrable.

Prove:

(1) $\int_0^{\frac{\pi}{2}} f(\cos x)\,dx = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx$;

(2) $\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\,dx$。

(1) $t = \frac{\pi}{2} - x$

$$\int_0^{\frac{\pi}{2}} f(\cos x)\,dx = \int_0^{\frac{\pi}{2}} f(\sin t)\,dt = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx \qquad \#.$$

(2) $t = \pi - x \qquad -\int_{\pi}^{0} \Leftrightarrow \int_0^{\pi}$

$$\underbrace{\int_0^{\pi} x f(\sin x)\,dx}_{I} = \int_{\pi}^{0} (\pi - t)\, f(\sin t)\,(-dt)$$

$$= \pi \int_0^{\pi} f(\sin x)\,dx - \underbrace{\int_0^{\pi} x f(\sin x)\,dx}_{I}$$

$$I = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\,dx \qquad \#$$

$$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx.$$

Use the results of previous question to find the following integrals.

(1) $\int_0^{\pi} x\sin^4 x\,dx$; (2) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$; $(\arctan x)' = \frac{1}{1+x^2}$

(3) $\int_0^{\pi} \frac{x}{1+\sin^2 x}dx$。

(1) $= \frac{\pi}{2}\int_0^{\pi} \sin^4 x\,dx$ $\qquad \int_0^{\pi}\sin^2 x\,dx = \frac{\pi}{2}.$

$= \frac{\pi}{2}\cdot\left\{\underbrace{-\left[\frac{\cos x\sin^3 x}{4}\right]_0^{\pi}}_{0} + \frac{3}{4}\int_0^{\pi}\sin^2 x\,dx\right.$

$= \frac{3\pi^2}{16}$

(2) $= \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$ $\qquad u = \cos x$

$= \frac{\pi}{2}\cdot\left(\underbrace{-\int_{-1}^{1} -\frac{1}{1+u^2}du}_{=\frac{\pi}{2}}\right)$

$= \frac{\pi^2}{4}$

(3) $= \frac{\pi}{2}\int_0^{\pi} \frac{dx}{1+\sin^2 x}$ $\qquad u = \tan x$

$= \frac{\pi}{2}\int_0^{\pi} \frac{du}{1+(\sqrt{2}\tan x)^2}$

$= \frac{\pi}{\sqrt{2}}\arctan(\sqrt{2}\tan x)\Big|_0^{\frac{\pi}{2}} = \frac{\sqrt{2}\pi^2}{4}$

Taylor series

More general, for $f(x)$ that is infinitely differentiable, we can approximate it by Taylor series

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

In above examples, we take $a = 0$ and $f(x)$ be $\sin x / \cos x$.

We can also think this strategy is an extension of linear approximation

$$f(x) \approx f(a) + f'(a)(x-a)$$

Maclaurin

When a=0,

Sin x = $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots$

Cos x = $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$

e^x = $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

arctan x = $x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1}$

$f(x) = e^x$. $x=0$,

$f(x) = f'(x) = f''(x) = \cdots = f^{(n)}(x) = e^x$,

$f(0) = f'(0) = \cdots = f^{(n)}(x) = e^0 = 1$.

$f(x) = e^x \approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots$

$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

$x - a = x - 0 = x$

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

Set / interval

Functions

Limits L'H ☆

Derivative C.R. ☆

Integral substitution

Int by parts

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