Calculus IB: Review

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Final Exam (Online)

1. Dec 15, 12:30pm - 15:30pm
2. HKUST CANVAS system with zoom proctoring
3. Click to here to see detailed regulations!

Final Exam (Online)

The following things will not be contained in final exam:

1. precise definition of limit
2. everything about (ε, δ)-definition
3. extended real number system (maybe useful to exam)
4. first order condition of convex function
5. gradient descent algorithm
6. convergence of Newton's method (but algorithm is required)
7. integration by parts (maybe useful to exam)

There are only multiple choice questions. Hence, the proof of theorems will not be checked, but you should remember the results.

Outline

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Notations of Sets

A set is a well-defined collection of distinct elements.

1. We can list all elements: e.g., the expression $\{2, 5, 7\}$ means a set consisting of three numbers: 2, 5 and 7.
2. Capital letters are often used to denote a set; e.g., $A = \{2, 5, 7\}$, where 2, 5, 7 are called the elements of the set A.
3. We use $\{x : P(x)\}$ to denote the set which is consisted of all elements x satisfying the description $P(x)$. For example:
   - $\{x : (x-2)(x-3) = 0\}$ is actually a set of two numbers: 2, 3
   - $\{x : (x-2)(x-3) > 0\}$ is the solution set of the inequality: $(x-2)(x-3) > 0$
   - $\{x : x \text{ is the square of an integer}\}$ is the set of 0, 1, 4, 9, 16, 25...

Notations of Intervals

Infinity, denoted by ∞, represents something that is larger than any real number. We use $-\infty$ to represent *negative infinity* that is smaller than any real number.

Let a and b be two real numbers. We define different classes of interval as follows.

Open Intervals	Closed Intervals
$(a, b) = \{x : a < x < b\}$	$[a, b] = \{x : a \leq x \leq b\}$
$(-\infty, a) = \{x : x < a\}$	$(-\infty, a] = \{x : x \leq a\}$
$(a, \infty) = \{x : x > a\}$	$[a, \infty) = \{x : x \geq a\}$

Half Open Half Closed Intervals
$[a, b) = \{x : a \leq x < b\}$
$(a, b] = \{x : a < x \leq b\}$

The interval $[a, b] = (a, b) = [a, b) = (a, b] = (a, a) = [a, a) = (a, a]$ contains nothing when $a > b$. We call it empty set, denoted by $\emptyset$ or $\{\}$.

The interval $(-\infty, \infty)$ formed by all real numbers, which is considered as both open and closed.

Solving Inequalities

For any real numbers a, b, and c,

1. if $a < b$, then $a + c < b + c$;
2. if $a < b$, then $a - c < b - c$;
3. if $a < b$ and $c > 0$, then $ac < bc$;
4. if $a < b$ and $c < 0$, then $ac > bc$;

Watch out when multiplying a negative number c on $a < b$, the result is $ac > bc$, rather than $ac < bc$!

Absolute Value

No matter what a mathematical expression $\blacksquare$, we have

$$|\blacksquare| = \begin{cases} \blacksquare & \text{if } \blacksquare \geq 0, \\ -\blacksquare & \text{if } \blacksquare < 0. \end{cases}$$

Note also that for any positive real number k, we have

1. $|\blacksquare| < k \iff -k < \blacksquare < k$
2. $|\blacksquare| > k \iff \blacksquare < -k$ or $\blacksquare > k$

Properties of Absolute Values

Some properties of absolute values:

1. $|-x| = |x|$
2. $|xy| = |x||y|$
3. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, where $y \neq 0$
4. $|x+y| \leq |x| + |y|$ (triangle inequality)

where equality holds if and only if x, y are of the same sign (equivalently $ab > 0$), or one of them is 0.

What is a Function?

- A *function* f is a rule that assigns to each element x in a set D exactly one element in a set E, which is denoted by $f(x)$ and called the *function value of f at x*.
- The set D is called the *domain of f* and the set E is called the *codomain of f*.
- A function f with domain D and codomain E is usually denoted by $f : D \longrightarrow E$.
- We can think of a function $f : D \longrightarrow E$ as an input-output machine which produces a unique output value $f(x)$ in the codomain E for any given input value x taken from the domain D.
- By considering the set of all function values of f, we have the *range* of the function: $\text{range of } f = \{f(x) : x \text{ is in the domain } D\}$.
- Note that the range of a function $f : D \longrightarrow E$ may not be the whole codomain E. f is said to be *onto* or *surjective* if $E = \text{range of } f$.

What is a Function?

- Given a function $y = f(x)$, the symbol x which represents numbers in the domain of f is called the *independent variable*, and the symbol y, which represents the function values in the range of f, is called the *dependent variable*.
- The *graph* of a function $f : D \longrightarrow E$ is just the set of ordered pairs of numbers

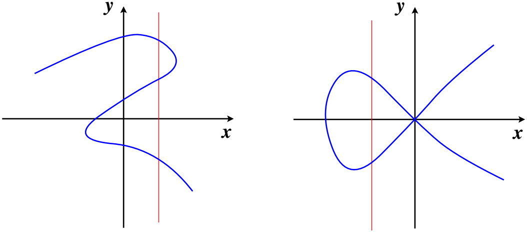
$$\text{graph of } f = \{(x, f(x)) : x \text{ is a number in } D\}$$

which can be geometrically plotted as a set of coordinate points in the xy-plane, if the function f is not too complicated.

Graph of a Function

Vertical line test for the graph of a function

- The graph of any function f should intersect every vertical line at most once (since for any number c in the domain of f, only one function value $f(c)$ is assigned).
- Conversely, any set of points in the xy-plane passing this test can be used to defined a function graphically.



These curves cannot be the graph of any function, since they fail the vertical line test

Some Elementary Functions

Following elementary mathematical functions you need to get familiar

- constant functions; e.g., 2, π, e.
- polynomial functions; e.g., $f(x) = x^3 + 2x^2 - 4x + 5$.
- rational functions; e.g., $f(x) = \dfrac{x^3 + 2x^2 - 4x + 5}{x^2 + 2x + 7}$.
- power functions; e.g., $f(x) = x^{3/2}$.
- exponential functions; e.g., $f(x) = 10^x$.
- logarithmic functions; e.g., $f(x) = \log_{10} x$.
- trigonometric functions; e.g., $\sin x$, $\cos x$, $\tan x$.
- inverse trigonometric functions; e.g., $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$.

Basic Operations: Sum, Product and Quotient

Given real-valued functions f and g, we can define new functions $f+g$ (*sum*), fg (*product*), and $\frac{f}{g}$ (*quotient*) simply by setting following rules:

$$(f+g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

as long as both function values, $f(x)$ and $g(x)$, are well-defined, and the corresponding arithmetic operations on them are valid.

However, we need to be careful with the domains of these functions.

Basic Operations: Sum, Product and Quotient

Domains of sum, product and quotient

1. For either $(f+g)(x)$ or $(fg)(x)$, the input value x must be in both the domain of f and the domain of g in order to have well-defined function values to add or to multiply. Hence the domain of $f+g$, or fg, is

$$\{x : x \text{ is is the domain of } f \text{ and } x \text{ is also in the domain of } g\}$$

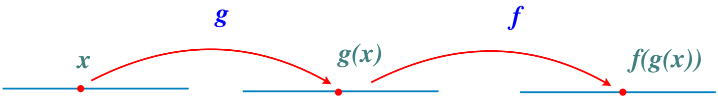
2. For $\dfrac{f(x)}{g(x)}$ to be well-defined, $f(x)$ and $g(x)$ have to be well-defined, and $g(x)$ has to be non-zero. Hence the domain of the function $\dfrac{f}{g}$ is

$$\{x : x \text{ is in the domain of } f, \text{ and } x \text{ is in the domain of } g, \text{ and } g(x) \neq 0\}$$

Basic Operations: Composition

One can also connect two "input-output machines" (functions) to form a new function, called the *composition* of f and g and denoted by the notation $f \circ g$, which is defined by

$$(f \circ g)(x) = f(g(x))$$



Obviously, we need $g(x)$ to be well-defined first, and then $g(x)$ to be in the domain of f in order to have a well-defined function value $f(g(x))$. Hence the domain of $f \circ g$ is given by

$$\begin{aligned} &\text{domain of } f \circ g \\ =&\{x : x \text{ is in the domain of } g \text{ and } g(x) \text{ is in the domain of } f\} \end{aligned}$$

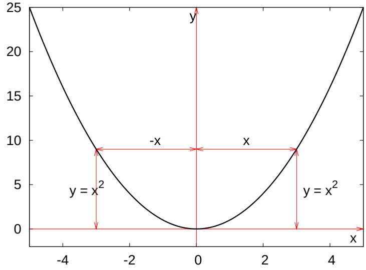
Even and Odd Functions

A function $y = f(x)$ is called an

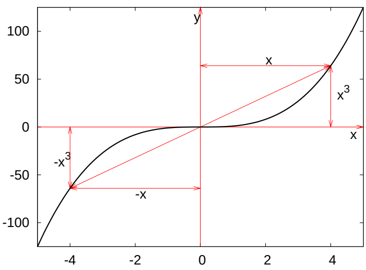
$$\begin{cases} \text{even function if } f(-x) = f(x) \\ \text{odd function if } f(-x) = -f(x) \end{cases}$$

for all x in the domain of f.

(a) $y = x^2$ is an even function



(b) $y = x^3$ is an odd function

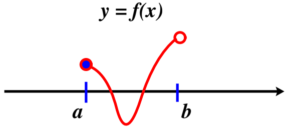


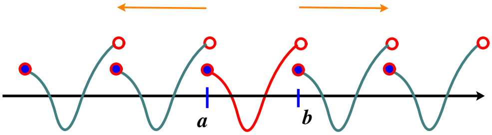
Periodic Functions

A function $f(x)$ is *periodic* if there is a number $T \neq 0$ such that $f(x+T) = f(x)$ for all x in the domain. The smallest such $T > 0$, if it exists, is called the *(fundamental) period* of the periodic function.

The graph of a periodic function does not change, if it is shifted to the left (or right), by a distance equal to an integral multiple of the period.

Any function f defined on the interval $[a, b)$ can be extended to a periodic function defined on the entire real line: keep shifting the graph by a distance of $b - a$.





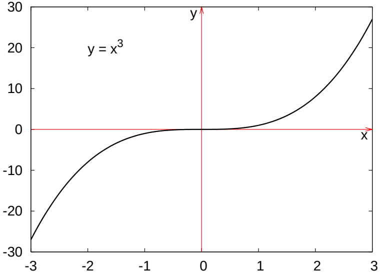
Increasing and Decreasing Functions

A function $y = f(x)$ is called

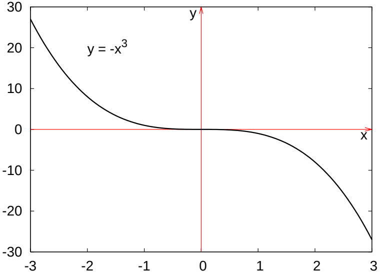
$$\begin{cases} \text{an increasing function} & \text{if } f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \\ \text{a decreasing function} & \text{if } f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \end{cases}$$

for all x_1, x_2 in the domain of f.

(a) increasing function: $y = x^3$



(b) decreasing function: $y = -x^3$



Transformations of Graphs

1. Graph of $y = f(x) + k$:
$$\begin{cases} \text{upward shifting of the graph of } f \text{ by } k \text{ units if } k > 0 \\ \text{downward shifting of the graph of } f \text{ by } k \text{ units if } k < 0 \end{cases}$$
2. Graph of $y = f(x + k)$:
$$\begin{cases} \text{shifting the graph of } f \text{ to the right by } |k| > 0 \text{ units if } k < 0 \\ \text{shifting the graph of } f \text{ to the left by } k \text{ units if } k > 0 \end{cases}$$
3. Graph of $y = -f(x)$: reflecting the graph of f across the x-axis.
4. Graph of $y = f(-x)$: reflecting the graph of f across the y-axis.
5. Graph of $y = kf(x)$, where $k > 0$:
$$\begin{cases} \text{stretching the graph of } f \text{ in } y\text{-direction by a factor of } k \text{ if } k > 1 \\ \text{compressing the graph of } f \text{ in } y\text{-direction by a factor of } k \text{ if } 0 < k < 1 \end{cases}$$
6. Graph of $y = f(kx)$, where $k > 0$:
$$\begin{cases} \text{compressing the graph of } f \text{ in } x\text{-direction by a factor of } k \text{ if } k > 1 \\ \text{stretching the graph of } f \text{ in } x\text{-direction by a factor of } k \text{ if } 0 < k < 1 \end{cases}$$

One-to-One Functions

1. A function f is said to be *one-to-one* if $f(x_1) \neq f(x_2)$ for *any* two numbers $x_1 \neq x_2$ in the domain of f.
2. In other words, $f(x)$ never takes on the same function value twice or more times when x runs through the domain of f; or equivalently, the equation
$$f(x) = b$$
has exactly one solution for any b in the range of f.

If f is a one-to-one function, then for any b in the range of f, the equation $f(x) = b$ has exactly one solution in the domain of f.

We can therefore define *inverse function* of f, usually denoted by f^{-1} (Warning: the symbol f^{-1} here does not mean $\frac{1}{f}$), by reversing the roles of the domain and range of f as follows:

$$f^{-1} : \begin{array}{c}\text{range of } f \\ \| \\ \text{domain of } f^{-1}\end{array} \longrightarrow \begin{array}{c}\text{domain of } f \\ \| \\ \text{range of } f^{-1}\end{array}$$

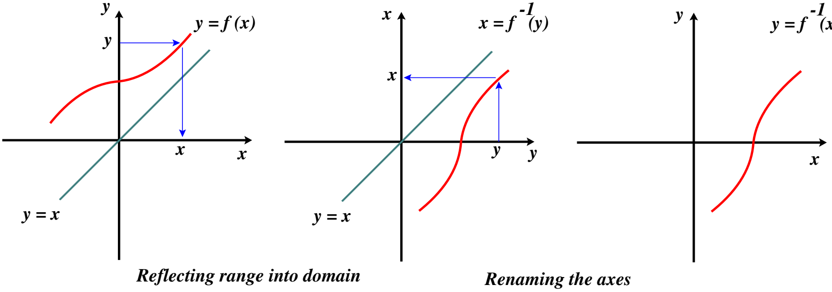
where

$$f^{-1}(b) = \text{the unique solution of the equation } f(x) = b$$

for any b in the domain of f^{-1} (i.e., the range of f).

Graphs of Inverse Functions

The graph of the inverse function $y = f^{-1}(x)$ can be obtained by reflecting the graph of the one-to-one function $y = f(x)$ across the line $y = x$, or simply by renaming the x-axis as the y-axis, and y-axis as the x-axis.



Power Functions

Note that for any positive integer n, the function $\frac{1}{x^n}$ can also be expressed in the form of power function as $\frac{1}{x^n} = x^{-n}$.

The *exponent laws* for integer powers (or exponents) then follow easily:

$$\text{(i)}\ \ x^0 = 1 \text{ (by convention)} \qquad \text{(ii)}\ \ x^{n+m} = x^n x^m \qquad \text{(iii)}\ \ x^{n-m} = \frac{x^n}{x^m}$$

$$\text{(iv)}\ \ (x^n)^m = x^{nm} \qquad \text{(v)}\ \ (xy)^n = x^n y^n \qquad \text{(vi)}\ \ \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

where n, m are any integers.

Exponential Functions

For any positive real number $a \neq 1$, the *exponential function with base a* is given by $y = a^x$.

1. The domain of $y = a^x$ is $(-\infty, \infty)$.
2. The range of $y = a^x$ is $(0, \infty)$.
3. We also have

$$y = a^x = \begin{cases} \text{is an increasing function} & \text{if } a > 1, \\ \text{is a decreasing function} & \text{if } 0 < a < 1. \end{cases}$$

Logarithmic Functions

An exponential function $y = a^x$ must be one-to-one (try to prove it), and hence has an inverse function, which is denoted by $x = \log_a y$, by reversing the roles of the domain and range:

$$\begin{cases} y = a^x \\ \text{domain: } -\infty < x < \infty \\ \text{range: } y > 0 \end{cases}$$

$$\longleftrightarrow \begin{cases} x = \log_a y \\ \text{domain: } y > 0 \\ \text{range: } -\infty < x < \infty \end{cases}$$

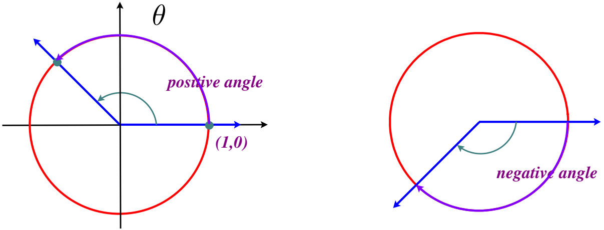
$$\longleftrightarrow \begin{cases} y = \log_a x \\ \text{domain: } x > 0 \\ \text{range: } -\infty < y < \infty \end{cases}$$

Properties of Exponential and Logarithmic Functions

Exponential Function	Logarithmic Function
$a^0 = 1$	$\log_a 1 = 0$
$a^1 = a$	$\log_a a = 1$
$a^x = a^x$	$\log_a a^x = x$
$a^{\log_a x} = x$	$\log_a x = \log_a x$
$a^x a^y = a^{x+y}$	$\log_a xy = \log_a x + \log_a y$
$\frac{a^x}{a^y} = a^{x-y}$	$\log_a \frac{x}{y} = \log_a x - \log_a y$
$(a^x)^y = a^{xy}$	$\log_a x^y = y \log_a x$
	$\log_c x = \frac{\log_a x}{\log_a c}$

Radian Measure of an Angle

If the point $(1, 0)$ starts to travel along the unit circle centered at the $(0, 0)$ through a distance θ in counterclockwise direction, the angle subtended by the corresponding circular arc is said to be a positive angle with *radian measure* θ. Angles obtained by clockwise rotations are considered as negative angles.



Directed angle : angle can be assigned a $+$ve or -ve sign

Radian Measure of an Angle

1. Recall that the length of a unit circle is 2π. Thus the radian measure of a 360° angle is 2π, and -2π if the angle is -360°.
2. In proportion, the degree measure and radian measure of an angle can be converted to each other according to

$$\frac{\text{radian measure}}{\text{degree measure}} = \frac{2\pi}{360} = \frac{\pi}{180}$$

3. The arc length and area of a circular section subtended by an angle θ in radians can be determined according to the following proportion:

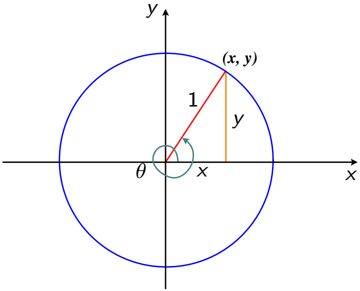
$$\frac{\text{circular sector area}}{\text{circle area}} = \frac{\theta}{2\pi} = \frac{\text{circular arc length}}{\text{circle length}}$$

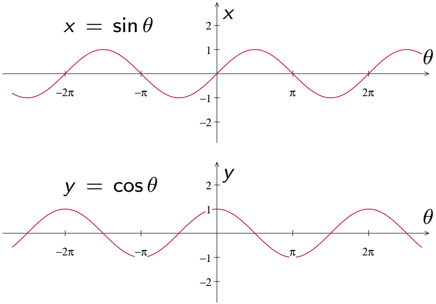
$$\frac{\text{circular sector area}}{\pi r^2} = \frac{\theta}{2\pi} = \frac{\text{circular arc length}}{2\pi r}$$

and $\text{circular sector area} = \frac{1}{2}r^2\theta$ and $\text{circular arc length} = r\theta$.

When a point originally at $(0, 1)$ moves along the unit circle through an angle of θ radians, the coordinates of the position (x, y) reached by the point depend on the value of θ, i.e., they are functions of θ:

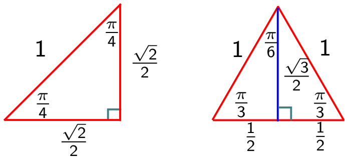
$$y = \sin\theta \quad \text{and} \quad x = \cos\theta, \quad \text{where } \theta \in (-\infty, +\infty) \text{ and } x, y \in [-1, 1].$$





Some Function Values of $\sin\theta$ and $\cos\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	-1



- $\sin\theta = 0$ if and only if $\theta = n\pi$ for some integer n. (points on the unit circle with zero y-coordinates are $(\pm 1, 0)$)
- $\cos\theta = 0$ if and only if $\theta = (2n+1)\frac{\pi}{2} = (n+\frac{1}{2})\pi$ for some integer n. (points on the unit circle with zero x-coordinates are $(0, \pm 1)$)

Properties of Sine and Cosine

- $\sin^2\theta + \cos^2\theta = 1$
- $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$
- $\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$
- $\sin(-\theta) = -\sin\theta$
- $\cos(-\theta) = \cos\theta$
- $\sin(\theta + \pi) = -\sin\theta$
- $\cos(\theta + \pi) = -\cos\theta$
- $\sin(\pi - \theta) = \sin\theta$
- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$
- $\cos(\pi - \theta) = -\cos\theta$
- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$

More Trigonometric Functions

Four other trigonometric functions, namely, $\tan\theta$ (*tangent*), $\cot\theta$ (*cotangent*), $\csc\theta$ (*cosecant*), and $\sec\theta$ (*secant*) are defined by

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	domain: $\{\theta : \cos\theta \neq 0\}$ range: $(-\infty, \infty)$
$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	domain: $\{\theta : \sin\theta \neq 0\}$ range: $(-\infty, \infty)$
$\csc\theta = \dfrac{1}{\sin\theta}$	domain: $\{\theta : \sin\theta \neq 0\}$ range: $(-\infty, -1] \cup [1, \infty)$
$\sec\theta = \dfrac{1}{\cos\theta}$	domain: $\{\theta : \cos\theta \neq 0\}$ range: $(-\infty, -1] \cup [1, \infty)$

We have the identities $1 + \tan^2\theta = \sec^2\theta$ and $1 + \cot^2\theta = \csc^2\theta$.

Trigonometric Identities: Angle Addition and Subtraction

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

Trigonometric Identities: Product to Sum/Sum to Product

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha+\beta) - \cos(\alpha-\beta)\right]$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

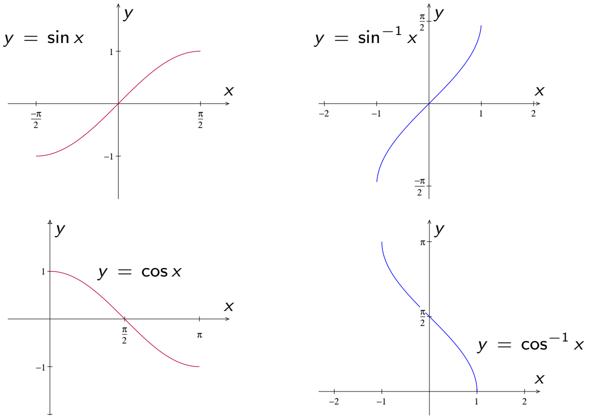
$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Recall that the graph of $y = \sin^{-1} x$ can be found by reflecting the part of the graph of $y = \sin x$, with $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, across the line $y = x$.

The inverse trigonometric functions $\cos^{-1} x$ can also be defined by inverting the functions $\cos x$ with domain restricted to $0 \leq x \leq \pi$.



Inverse trigonometric functions as solutions of trigonometric equations

- $\sin^{-1} x$ is the unique solution θ (angle in radian measure) of the equation $x = \sin\theta$ chosen within the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (solvable for any $-1 \le x \le 1$).
- $\cos^{-1} x$ is the unique solution θ (angle in radian measure) of the equation $x = \cos\theta$ chosen within the closed interval $[0, \pi]$ (solvable for any $-1 \le x \le 1$).
- $\tan^{-1} \theta$ is the unique solution θ (angle in radian measure) of the equation $x = \tan\theta$ chosen within the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (solvable for any $-\infty < x < \infty$).

General Solution of Trigonometric Equations

Using the inverse trigonometric functions, one can express the general solutions of some basic trigonometric equations as follows:

$$\sin x = a \quad \begin{cases} x = n\pi + (-1)^n \sin^{-1} a & \text{if} \quad -1 < a < 1 \\ x = 2n\pi + \frac{\pi}{2} & \text{if} \quad a = 1 \\ x = 2n\pi - \frac{\pi}{2} & \text{if} \quad a = -1 \\ \text{no solution} & \text{if} \quad |a| > 1 \end{cases}$$

$$\cos x = a \quad \begin{cases} x = 2n\pi \pm \cos^{-1} a & \text{if} \quad -1 \le a \le 1 \\ \text{no solution} & \text{if} \quad |a| > 1 \end{cases}$$

$$\tan x = a \qquad x = n\pi + \tan^{-1} a \qquad \text{for any real number } a$$

where $n = 0, \pm1, \pm2, \pm3, \cdots$ goes through the set of all integers.

Outline

The Slope of a Tangent Line

In geometry, the *tangent line* to a curve at a given point is the straight line that "just touches" the curve at that point.

The *secant line* of a curve is a line that intersects the curve at a minimum of two distinct points.

Recall that the slope of a straight line passing through two distinct points (x_1, y_1), (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

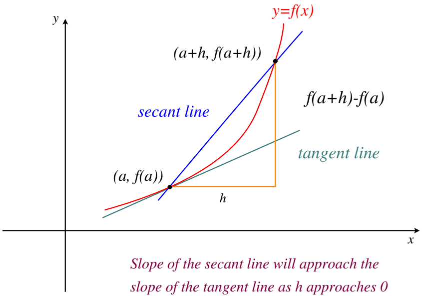
We can get better and better approximation of the slope $m_{\tan}$ at $(x_0, f(x_0))$ by looking at slope of secant line through $(x_0, f(x_0))$ and $(x_0 + h, f(x_0 + h))$ on the graph, when h is chosen to be closer and closer to 0.

Limit Definition of Derivative

In general, given a function f, we can consider the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$ in a similar manner by looking at limiting behavior of the slopes of nearby secant lines:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \triangleq f'(a) \text{ , whenever the limit exists.}$$

$f'(a)$ is called the *derivative of f at a*.



Examples of Derivative

The term

$$\frac{f(a+h)-f(a)}{h}$$

is usually considered as the *average rate of change* of the function values of f over the interval $[a, a+h]$, and hence the limit $f'(a)$ is considered as the *instantaneous rate of change* of f at a.

Intuitively speaking, given real numbers c and L, the expression

$$\lim_{x \to c} f(x) = L$$

means that $f(x)$ becomes arbitrarily close to L as x approaches c. We allow c or L be ∞ or $-\infty$.

The precise meanings of "arbitrarily close" and "approaches" require (ε, δ)-definition which is not required in exam.

Limit and Natural Logarithmic Function

One condition that determines the number e, which is the *base of the natural logarithmic function*, is that the slope of the tangent line to the graph of the natural logarithmic function $y = \log_e x = \ln x$ at $(1, 0)$ is 1.

Using the limit notation, e is the number which satisfies

$$\lim_{h \to 0} \log_e (1+h)^{\frac{1}{h}} = 1.$$

Since we have $\log_e e = 1$, one way to define the number e is

$$e = \lim_{h \to 0} (1+h)^{\frac{1}{h}} \approx 2.7182818 \cdots$$

We also have

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1, \quad \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} = e^x$$

Limits of Function Values and One-Side Limits

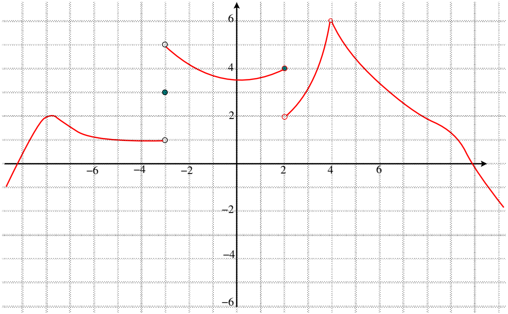
An important point to keep in mind is that finding $\lim_{x\to a} f(x)$ is NOT the same as finding the function value $f(a)$.

1. $\lim_{x\to a} f(x)$ may exist even if $f(x)$ is undefined at $x = a$
2. $\lim_{x\to a} f(x)$ may not exists even if $f(x)$ is well-defined at $x = a$

The limit $\lim_{x\to a} f(x)$ exists and equals the value L if and only if the two one-sided limits exist, and are equal to L:

$$\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x).$$

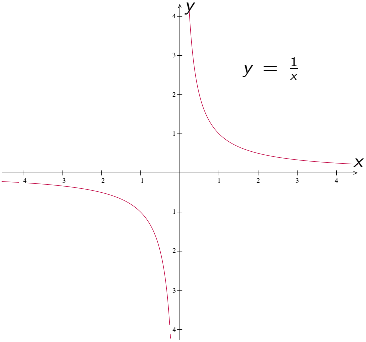
Finding Limits by Graphs



- $\lim_{x \to 4} f(x) = 6$, while $f(4)$ is not well-defined.
- $f(-3) = 3$, but the left-hand limit $\lim_{x \to -3^-} f(x) = 1$ and the right-hand limit $\lim_{x \to -3^+} f(x) = 5$. Hence, the $\lim_{x \to -3} f(x)$ is does not exist!
- $\lim_{x \to 2^-} f(x) = 4 = f(2)$, but $\lim_{x \to 2^+} f(x) = 2 \neq f(2) = 4$. The (two-sided) limit $\lim_{x \to 2} f(x)$ does not exist!

(a) $\lim_{x \to 0^+} \frac{1}{x} = +\infty$ (b) $\lim_{x \to 0^-} \frac{1}{x} = -\infty$ (c) $\lim_{x \to +\infty} \frac{1}{x} = 0$

(d) $\lim_{x \to -\infty} \frac{1}{x} = 0$ (e) $\lim_{x \to a} \frac{1}{x} = \frac{1}{a}$ for all real number $a \neq 0$



The line $y = 0$ (x-axis) is called a *horizontal asymptote* of the function $f(x) = \frac{1}{x}$.
The line $x = 0$ (y-axis) is called a *vertical asymptote* of this function.

Horizontal Asymptote and Vertical Asymptote

In general, we may consider the limiting behavior of $f(x)$ as $x \to \infty$ or $x \to -\infty$, or consider some one-sided limits to see if $f(x)$ is approaching ∞ or $-\infty$ as $x \to a^+$ or $a \to a^-$.

1. $y = L$ is a *horizontal asymptote* of the function $f(x)$ if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.
2. $x = b$ is a *vertical asymptote* of the function $f(x)$ if at least one of the following holds:

   a) $\lim_{x\to b^-} f(x) = \infty$, b) $\lim_{x\to b^-} f(x) = -\infty$,
   c) $\lim_{x\to b^+} f(x) = \infty$, d) $\lim_{x\to b^+} f(x) = -\infty$.
3. A function could has two different horizontal asymptotes $y = L_1$ and $y = L_2$ if $\lim_{x\to\infty} f(x) = L_1 \neq \lim_{x\to-\infty} f(x) = L_2$.
4. In any case, a function can have at most two horizontal asymptotes.

Vertical Asymptote and Slant Asymptote

1. Given a function of the form

$$\frac{f(x)}{g(x)},$$

the vertical line defined by $x = a$ is a vertical asymptote as long as $f(a) \neq 0$ but $\lim_{x \to a^-} g(x) = 0$ or $\lim_{x \to a^+} g(x) = 0$. (Note that we do NOT require $f(a)$ is well-defined, but $f(a) \neq 0$ if it is well-defined.)

2. If $f(x) = ax + b + g(x)$ with $g(x) \to 0$ as $x \to \infty$ or $x \to -\infty$, then the straightline given by $y = ax + b$ is called a *slant asymptote* of f.

Some Useful Limit Laws

Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists on (extended) real numbers, then we have:

1. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$ for any constant c
2. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}[f(x)]^p = \left(\lim_{x\to a} f(x)\right)^p$ for any rational exponent p when $(\lim_{x\to a} f(x))^p$ exists.

Extended Real Number System

We introduce extended real number system to address the calculation contains the ∞ and $-\infty$. It is useful in describing the algebra on infinities and the various limiting behaviors in calculus.

Recall that $\mathbb{R} = (-\infty, \infty)$ presents the set of all real number.

The extended real number system is denoted by $\overline{\mathbb{R}}$ or $[-\infty, +\infty]$ or $\mathbb{R} \cup \{-\infty, +\infty\}$.

Here, "$+\infty$" is equivalent to "∞" and "$-(-\infty)$".

Arithmetic Operations on $\overline{\mathbb{R}}$

$$a + \infty = +\infty + a = +\infty, \qquad a \neq -\infty$$

$$a - \infty = -\infty + a = -\infty, \qquad a \neq +\infty$$

$$a \cdot (+\infty) = +\infty \cdot a = +\infty, \qquad a \in (0, +\infty]$$

$$a \cdot (-\infty) = -\infty \cdot a = -\infty, \qquad a \in (0, +\infty]$$

$$a \cdot (+\infty) = +\infty \cdot a = -\infty, \qquad a \in [-\infty, 0)$$

$$a \cdot (-\infty) = -\infty \cdot a = +\infty, \qquad a \in [-\infty, 0)$$

Arithmetic Operations on $\overline{\mathbb{R}}$

$$\frac{a}{+\infty} = \frac{a}{-\infty} = 0, \qquad a \in \mathbb{R}$$

$$\frac{+\infty}{a} = +\infty, \qquad a \in (0, +\infty)$$

$$\frac{-\infty}{a} = -\infty, \qquad a \in (0, +\infty)$$

$$\frac{+\infty}{a} = -\infty, \qquad a \in (-\infty, 0)$$

$$\frac{-\infty}{a} = +\infty, \qquad a \in (-\infty, 0)$$

Arithmetic Operations on $\overline{\mathbb{R}}$

$$
\begin{aligned}
a^{+\infty} &= +\infty & a &\in (1, +\infty] \\
a^{-\infty} &= 0 & a &\in (1, +\infty] \\
a^{+\infty} &= 0 & a &\in [0, 1) \\
a^{-\infty} &= +\infty & a &\in [0, 1) \\
0^{a} &= 0 & a &\in (0, +\infty] \\
(+\infty)^{a} &= +\infty & a &\in (0, +\infty] \\
(+\infty)^{a} &= 0 & a &\in [-\infty, 0)
\end{aligned}
$$

Extended Real Number System

However, the following expressions are still undefined

$$\frac{+\infty}{+\infty} \quad \frac{+\infty}{-\infty} \quad \frac{-\infty}{+\infty} \quad \frac{-\infty}{-\infty}$$

$$0 \cdot (+\infty) \quad 0 \cdot (-\infty) \quad (+\infty) \cdot 0 \quad (-\infty) \cdot 0$$

$$\infty - \infty \quad (-\infty) - (-\infty)$$

The expression $1/0$ (or $0^a = +\infty$ for $a \in [-\infty, 0)$) is still left undefined, since

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \neq -\infty = \lim_{x \to 0^-} \frac{1}{x}.$$

The expression 0^a is undefined when $a \in [-\infty, 0)$, just like $1/0$.

Squeeze Theorem

Squeeze Theorem (or Sandwich Theorem)

Let I be an interval having the point a. Let g, f, and h be functions defined on I, except possibly at a itself. Suppose that for every x in I NOT equal to a, we have If $g(x) \leq f(x) \leq h(x)$ for all x near a, except perhaps when $x = a$, then

$$\lim_{x \to a} g(x) \leq \lim_{x \to a} f(x) \leq \lim_{x \to a} h(x)$$

whenever these limits exist. (The same is true for one-sided limits.)

We can prove the following classical result by using squeeze theorem

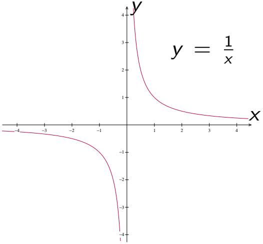
$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

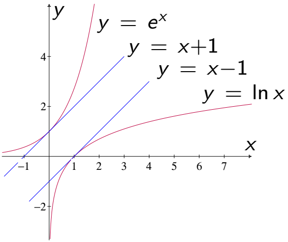
We also have the extended version

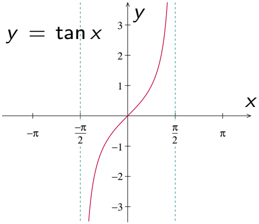
$$\lim_{x \to 0} \frac{\sin ax}{ax} = 1,$$

where $a \neq 0$ is a constant.

Some Useful Limits







$$\lim_{x\to\infty}\frac{1}{x}=0 \qquad \lim_{x\to\infty}e^x=\infty \qquad \lim_{x\to\frac{\pi}{2}^-}\tan x=\infty$$

$$\lim_{x\to-\infty}\frac{1}{x}=0 \qquad \lim_{x\to-\infty}e^x=0 \qquad \lim_{x\to-\frac{\pi}{2}^+}\tan x=-\infty$$

$$\lim_{x\to0^+}\frac{1}{x}=\infty \qquad \lim_{x\to\infty}\ln x=\infty \qquad \lim_{x\to\infty}\tan^{-1}x=\frac{\pi}{2}$$

$$\lim_{x\to0^-}\frac{1}{x}=-\infty \qquad \lim_{x\to0^+}\ln x=-\infty \qquad \lim_{x\to-\infty}\tan^{-1}x=-\frac{\pi}{2}$$

Continuity of Functions

If $f(c)$, $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ are well-defined and equal on real numbers (we do not consider ∞ or $-\infty$), we say that the function is *continuous* at $x = c$.

Sometimes, d is called a *point of discontinuity* of a function f if the condition $\lim_{x \to a} f(x) = f(d)$is not satisfied.

In this course, we focus on the continuity of functions defined on an interval, or the union of several intervals.

- If c is a real number in the domain of a function f such that a small open interval $(c - h, c + h)$ containing c, where $h > 0$, is entirely in the domain of f, c is called an *interior point* of the domain of f.
- A function $y = f(x)$ is said to be *continuous at an interior point* c in its domain if $\lim_{x \to c} f(x) = f(c)$.

Properties of Continuous Functions

- Sums, differences, products of continuous functions are continuous.
- If two functions $f(x)$, $g(x)$ are continuous at $x = c$ and $g(c) \neq 0$, then the quotient $\frac{f}{g}$ is continuous at $x = c$.
- Note also that if f is continuous at c and g is continuous at $f(c)$, then the composition of the two functions $g \circ f$ is continuous at c.

Properties of Continuous Functions

- The elementary functions $\sin x$, $\cos x$, $\tan x$, a^x and $\log_a x$ are all continuous at any point in their domains.
- Polynomial functions are continuous on real numbers.
- For any positive integer n, the root function $f^{1/n}$ of a function f continuous at $x = c$ is also continuous at $x = c$, as long as the power function is well-defined on an open interval containing c.
- Rational functions are continuous on the real line, except at the zeros of their denominators, i.e., continuous on their domains. Recall here that a *rational function* is a function of the form

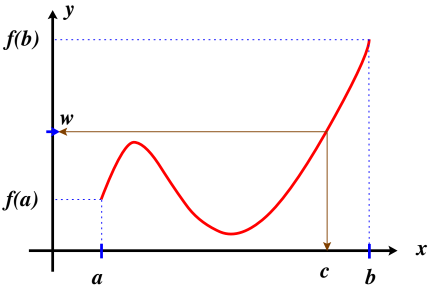
$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x), q(x)$ are polynomials with $q(x) \neq 0$.

Intermediate Value Theorem

Theorem (Intermediate Value Theorem)

Suppose the function $y = f(x)$ is continuous on a closed interval $[a, b]$ and let w be a real number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there must be a number c in (a, b) such that $f(c) = w$.



In other words, the equation $f(x) = w$ must have at least one root in the interval (a, b). The Intermediate Value Theorem is very useful in locating roots of equations.

Bisection Method

Example

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ in the interval $(1, 2)$.

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$, which is continuous on [1,2]. Then 0 is a number between $f(1)$ and $f(2)$:

$$-1 = f(1) < 0 < f(2) = 12.$$

By the Intermediate Value Theorem, there must be a number c in $(1, 2)$ such that $f(c) = 0$.

Similarly, $f(1.5) = 3.4 > 0$, hence the equation must have a root in the interval $(1, 1.5)$. We can also compute $f(1.25)$ to determine the root lies in $(1, 1.25)$ or $(1.25, 1.5)$.

Continuing in this manner, one can end up with the "Bisection Method" for locating approximate roots of equations.

Outline

Limit Definition of Derivatives

Recall that the rate of change of a function $y = f(x)$ at $x = a$ is a certain limit called the *derivative of f at a*, which is denoted by $f'(a)$, and is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \overset{or}{=} \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

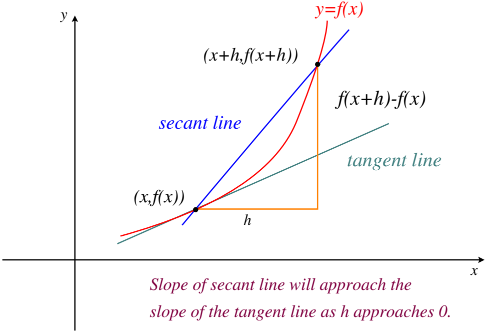
whenever the limit exists.

- The function f is said to be *differentiable at $x = a$* when $f'(a)$ exists on real numbers. (only correct for single variable calculus)
- Recall also that the limit $f'(a)$ can be interpreted as the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.

If we want to measure how fast the function value $y = f(x)$ changes as x varies, we consider the *derivative function* $f'(x)$, which is defined as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

whenever the limit exists. Geometrically speaking, f' is the slope function of f.



Limit Definition of Derivatives

Some other often used notations to denote the derivative $f'(x)$ of the function $y = f(x)$ are as follows:

$$\frac{df}{dx}, \quad \frac{dy}{dx}, \quad y', \quad \text{and} \quad \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a} = y'(a) = f'(a).$$

The process of finding the derivative of a given function is called differentiation.

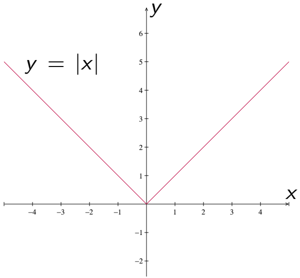
When computing derivatives by using the limit definition of derivative, it is sometimes called differentiating by the first principle.

Theorem

If f is differentiable at a point $x = a$, then f is continuous at $x = a$.

The derivative of a continuous function may not exist at some point.

A basic example is $f(x) = |x|$. Its derivative at $x = 0$, namely $f'(0)$, does not exist since there is no tangent line to the graph at $(0, 0)$.



Basic Formulas of Derivatives

Here are the derivatives of some elementary functions, which are the results of some limit computations.

1. $\dfrac{dc}{dx} = 0$, for any constant c
2. $\dfrac{dx^p}{dx} = px^{p-1}$, for any constant p
3. $\dfrac{de^x}{dx} = e^x, \qquad \dfrac{d\ln x}{dx} = \dfrac{1}{x}$
4. $\dfrac{d\sin x}{dx} = \cos x, \qquad \dfrac{d\cos x}{dx} = -\sin x$

Rules of Differentiation

Theorem

Suppose function f satisfies $\lim_{y \to x_0} f(y) = u_0$ and function g is continuous at u_0, then the composition function $(g \circ f)(y) = g(f(y))$ holds that

$$\lim_{y \to y_0} (g \circ f)(y) = \lim_{u \to u_0} g(u) = g(u_0).$$

Whenever f' and g' both exist, we have the following rules:

1. $\frac{d}{dx}(af + bg) = a\frac{df}{dx} + b\frac{dg}{dx} = af' + bg'$ for any constants a and b.
2. **Product Rule:** $\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx} = fg' + gf'$
3. **Quotient Rule:** $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} = \frac{gf' - fg'}{g^2}$

The Chain Rule

Let F is compositions of two functions f and g:

$$F(x) = (f \circ g)(x) = f(g(x)),$$

such that

1. g is a differentiable at x (the derivative $g'(x)$ exists),
2. and f is a is differentiable at $g(x)$ (the derivative $f'(g(x))$ exists);

then $y = F(x) = (f \circ g)(x)$ is differentiable at x, and its derivative is

$$F'(x) = f'(g(x)) \cdot g'(x).$$

Recite derivatives of $\sin x$ and $\cos x$, then show the others by quotient rule.

$$\frac{d\sin x}{dx} = \cos x \qquad \frac{d\cos x}{dx} = -\sin x \qquad \frac{d\tan x}{dx} = \sec^2 x$$

$$\frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

$$\frac{d\cot x}{dx} = -\csc^2 x \qquad \frac{d\sec x}{dx} = \sec x\tan x \qquad \frac{d\csc x}{dx} = -\csc x\cot x$$

Derivatives of Inverse Functions

Theorem (Derivatives of Inverse Function)

Suppose f is a differentiable and has inverse function f^{-1} over an interval I and x is a point in I such that $x = f(a)$ and $f'(a) \neq 0$, then f^{-1} is differentiable at x and its derivative is

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(x))}.$$

$$\frac{d}{dx}\left(\sin^{-1}x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\left(\cos^{-1}x\right) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1}x\right) = \frac{1}{1+x^2} \qquad \frac{d}{dx}\left(\cot^{-1}x\right) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\left(\sec^{-1}x\right) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}\left(\csc^{-1}x\right) = -\frac{1}{|x|\sqrt{x^2-1}}$$

In general, it is difficult or impossible to find the explicit expression of $y = f(x)$ by $F(x, y)$, but we can express $y' = f'(x)$ by x and y.

We desire to find $f'(x)$ directly from the implicit form $F(x, y) = 0$ without solving $y = f(x)$.

Implicit differentiation can be done as follows:

$$F(x, y) = 0 \quad \xrightarrow{\frac{d}{dx}\text{ both sides}} \quad \text{an equation to solve for } \frac{dy}{dx}$$

Chain Rule Version of Basic Derivative Formulas

The following chain rule versions of basic derivative formulas are convenient to use for calculation of derivatives.

$$\frac{d\blacksquare^p}{dx} = p\blacksquare^{p-1} \cdot \frac{d\blacksquare}{dx} \qquad \frac{de^{\blacksquare}}{dx} = e^{\blacksquare} \cdot \frac{d\blacksquare}{dx}$$

$$\frac{d\ln\blacksquare}{dx} = \frac{1}{\blacksquare} \cdot \frac{d\blacksquare}{dx} \qquad \frac{d\sin\blacksquare}{dx} = \cos\blacksquare \cdot \frac{d\blacksquare}{dx}$$

$$\frac{d\cos\blacksquare}{dx} = -\sin\blacksquare \cdot \frac{d\blacksquare}{dx} \qquad \frac{d\tan\blacksquare}{dx} = \sec^2\blacksquare \cdot \frac{d\blacksquare}{dx}$$

$$\frac{d\sec\blacksquare}{dx} = \sec\blacksquare \cdot \tan\blacksquare \cdot \frac{d\blacksquare}{dx} \qquad \frac{d\sin^{-1}\blacksquare}{dx} = \frac{1}{\sqrt{1-\blacksquare^2}} \cdot \frac{d\blacksquare}{dx}$$

$$\frac{d\tan^{-1}\blacksquare}{dx} = \frac{1}{1+\blacksquare^2} \cdot \frac{d\blacksquare}{dx}$$

Second Order Derivative (Second Derivative)

If $s = s(t)$ is the position function of a particle moving along a line represented by the x axis, then

$$\frac{dx}{dt} = \text{ velocity function } = v(t)$$

$$\frac{dv}{dt} = \text{ acceleration function} = a(t)$$

In particular, if m is the mass of the particle, and F is the force acting on the particle, Newton's Second Law $F = ma$ can be expressed as

$$F = m\frac{dv}{dt} = m\frac{d^2s}{dt^2}$$

where the second derivative means "the derivative of the derivative":

$$s''(t) = \frac{d^2s}{dt^2} \overset{\text{means}}{=} \frac{d}{dt}\left(\frac{ds}{dt}\right).$$

Higher Order Derivatives

The second order derivative of f is the derivative of the derivative of f:

$$\frac{d^2 f(x)}{dx^2} = f''(x) = (f')'(x).$$

The third order derivative of f is the derivative of the second order derivative of f:

$$\frac{d^3 f(x)}{dx^3} = f'''(x) = (f'')'(x).$$

In general, the n-th order derivative of f is the derivative of the $(n-1)$-th order derivative of f:

$$\frac{d^n f(x)}{dx^n} = f^{(n)}(x) = \left(f^{(n-1)}\right)'(x).$$

Outline

Rates of Change

When a function $y = f(x)$ describes the relation between two quantities represented by x and y respectively, the derivative function

$$f'(x) \quad \text{or} \quad \frac{dy}{dx}$$

is considers as the *rate of change* of the quantity y with respect to the quantity x.

Related Rates

The main idea about related rates is essentially the following.

Given some quantities

$$\left.\begin{array}{l} q_1 = q_1(t) \\ q_2 = q_2(t) \\ \vdots \\ q_n = q_n(t) \end{array}\right\} \quad \begin{array}{c}\text{which are all functions of } t, \\ \text{where } t \text{ may represent time or some other quantity,}\end{array}$$

if there is an equation relating all these quantities, then

$$\frac{d}{dt} \text{ of both sides of the relation}$$

$$\xrightarrow{gives} \quad \text{an equation relating the rates of changes } \frac{dq_1}{dt}, \frac{dq_2}{dt}, \ldots, \frac{dq_n}{dt}$$

Extreme Values of a Function

When studying a function, we sometimes need to determine its largest function value or smallest function value.

Suppose we have a function f, and c is a real number in its domain D.

- $f(c)$ is called the *global maximum* (or *absolute maximum*) of f on D if $f(c) \geq f(x)$ for *all* real number x in D.
- $f(c)$ is called the *global minimum* (or *absolute minimum*) of f on D if $f(c) \leq f(x)$ for *all* real numbers x in D.
- $f(c)$ is called a *local maximum* (or *relative maximum*) of f on D if $f(c) \geq f(x)$ for *numbers x in D which are "near" c*.
- $f(c)$ is called a *local minimum* (or *relative minimum*) of f on D if $f(c) \leq f(x)$ for *numbers x in D which are "near" c*.
- An *extremum* (or extreme value) is either a maximum or minimum, absolute or local.

Theorem (Extreme Value Theorem)

If f is continuous on a closed interval, then f attains an global maximum $f(c)$ and an global minimum $f(d)$ at some numbers c and d in $[a, b]$.

The global maximum/minimum may be reached at the boundary points of the closed interval $[a, b]$, or at points inside the open interval (a, b).

If $f(c)$ is a local maximum/minimum for some c in (a, b) and $f'(c)$ exists, then $f'(c) = 0$.

A number c in the domain of f is called a *critical number* or *critical point* if either $f'(c) = 0$ or $f'(c)$ does not exist.

Finding Global Maximum/Minimum

Theorem (Fermat's Theorem)

If f has a local maximum or local minimum at an interior point c, and if $f'(c)$ exists, then $f'(c) = 0$.

As a result, we obtain a basic approach to find the global maximum and minimum of a differentiable function f on a closed interval $[a, b]$ is:

1. Find all critical points of f in (a, b), and the respective function values.
2. Find the function values of f at the boundary points of the interval $[a, b]$.
3. Just compare these function values above to find the largest (global maximum) and smallest (global minimum).

Rolle's Theorem and Mean Value Theorem

Combining the extreme value theorem and Fermat's theorem, it is easy to conclude Rolle's theorem.

Theorem (Rolle's Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and $f(a) = f(b)$ and $a < b$, then $f'(c) = 0$ for some number $c \in (a, b)$.

Theorem (Mean Value Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some $c \in (a, b)$, or equivalently $f(b) - f(a) = f'(c)(b - a)$.

Here are some consequences of the mean value theorem:

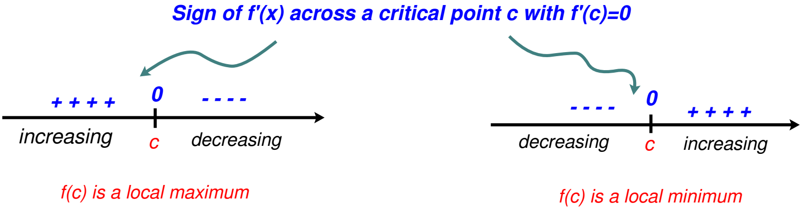
- If $f' = 0$ on the whole interval (a, b), then f is a constant function on the interval. (for any $a < x_1 < x_2 < b$, we have $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0$, i.e., $f(x_1) = f(x_2)$)
- If $f'(x) > 0$ for all x in an interval (a, b), then $f(x)$ is an increasing function on (a, b). (for any $a < x_1 < x_2 < b$, we have $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$, for some c between x_1 and x_2, i.e., $f(x_2) > f(x_1)$ since $f'(c) > 0$)
- If $f'(x) < 0$ for all x in an interval (a, b), then $f(x)$ is an decreasing function on (a, b).

Using 1st and 2nd Derivatives in Graphing

A lot about function $y = f(x)$ can be found by $f'(x)$ and $f''(x)$.

$$f(x) \begin{cases} f'(x) \begin{cases} \text{Critical Points : } f'(x) = 0 \text{ (or undefined)} \\ \text{Intervals of Increase/Decrease } (\pm \text{ sign of } f'(x)) \\ \text{1st Derivative Test for Local Extrema} \\ \text{(i.e., look at the sign line of } f'.) \end{cases} \\ f''(x) \begin{cases} \text{Convex/Concave Intervals } (\pm \text{sign of } f''(x)) \\ \text{Inflection Points - where concavity changes.} \\ \text{2nd Derivative Test for Local Extrema} \end{cases} \end{cases}$$

First Derivative Test



$f(c)$ is neither a local maximum nor a local minimum if the sign of f' does not change across c.

Second Derivative Test

Suppose f, f' and f'' are well defined on (a, b) and c in (a, b). Note that sufficient condition and necessary condition of local extrema are different.

- $f'(c) = 0$ and $f''(c) > 0$ mean $f(c)$ is a local minimum
- $f(c)$ is a local minimum means $f'(c) = 0$ and $f''(c) \geq 0$
- $f'(c) = 0$ and $f''(c) < 0$ mean $f(c)$ is a local maximum
- $f(c)$ is a local maximum means $f'(c) = 0$ and $f''(c) \leq 0$

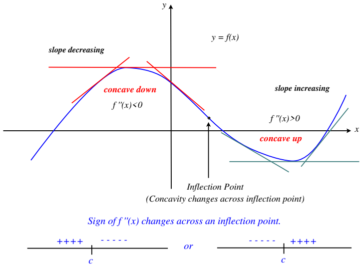
Please note that

- $f'(c) = 0$ and $f''(c) \geq 0$ does not mean $f(c)$ is a local minimum
- $f'(c) = 0$ and $f''(c) \leq 0$ does not mean $f(c)$ is a local maximum
- Consider the function $f(X) = x^3$ at $x = 0$.

Convexity/Concavity and 2nd Derivatives

What does the graph of $y = f(x)$ on an interval mean by the sign of f''?

- $f'' > 0 \implies f'$ is increasing (the slope of tangent line is increasing) $\implies f$ is concave up (or strictly convex)
- $f'' < 0 \implies f'$ is decreasing (the slope of tangent line is decreasing). $\implies f$ is concave down (or strictly concave)
- If concavity (up/down) on both sides of a point $(c, f(c))$ on the graph of the function $y = f(x)$, where f is continuous, are different, then the point is called a point of inflection.

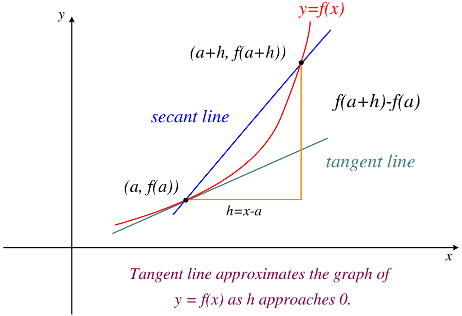


Linear Approximation

The *tangent line approximation at $x = a$*, or *linear approximation at $x = a$*, or *linearization at $x = a$*, of a function $y = f(x)$ (differentiable at $x = a$) is that we are using the tangent line equation (or the corresponding linear function) to approximate the given function.

$$y = f(x) \overset{\approx}{\longleftarrow} \text{Tangent Line Equation : } y = f(a) + f'(a)(x-a)$$

$$\Longrightarrow f(x) \approx f(a) + f'(a)(x-a) \qquad \text{for } x - a \approx 0$$



Tangent line approximates the graph of
y = f(x) as h approaches 0.

Differential of the Function

The tangent line approximation at x is

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x,$$

where Δx denotes some increment in x (which could be negative).

Then we use Δy or Δf to denote the change in the function values

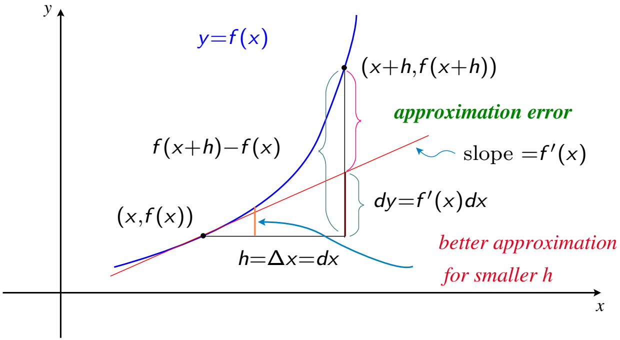
$$\Delta y = \Delta f = f(x + \Delta) - f(x).$$

and the linear approximation be expressed as

$$\Delta f \approx f'(x)\Delta x.$$

Note that $f'(x)\Delta x$ is the change of y-value along the tangent line!

Differential of the Function



The notation of differentials $df = f'(x)dx$ is obtained by expressing Δx as dx, and $dy = df = f'(x)dx$ can be used as an approximation of

$$\Delta y = f(x + \Delta x) - f(x).$$

Baby L'Hôpital's Rule

Theorem (Baby L'Hôpital's Rule, $\frac{0}{0}$-type)

Let $f(x)$ and $g(x)$ be continuous functions on an interval containing $x = a$, with $f(a) = g(a) = 0$. Suppose that f and g are differentiable, and f' and g' are continuous. Finally, suppose that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}.$$

We also have

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

and

$$\lim_{x \to a^-} \frac{f(x)}{g(x)} = \lim_{x \to a^-} \frac{f'(x)}{g'(x)}.$$

Theorem (Macho L'Hôpital's Rule, one-side)

Suppose that f and g are continuous on a closed interval $[a, b]$, and are differentiable on the open interval (a, b). Suppose that $g'(x)$ is never zero on (a, b) and $\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ exists, and that $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$. Then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$$

The previous versions apply to forms of type $\frac{\infty}{\infty}$ as well as $\frac{0}{0}$, and apply to limits as $x \to \infty$ or $x \to -\infty$ as well as to limits $x \to a^+$ or $x \to a^-$. In all of these cases, the rule is:

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}.$$

L'Hôpital's Rule

1. L'Hôpital's rule help us compute limits of the $\frac{0}{0}$ or $\frac{\infty}{\infty}$-type
2. L'Hôpital's rule is not a universal tool.
3. We must check the form of limit before applying L'Hôpital's rule.
4. Sometimes, simplifying the expression is more useful.

Newton's Method

Newton's method is a simple usage of the tangent lines in finding approximate solutions of a non-linear equation

$$f(x) = 0,$$

where f is differentiable and defined on real numbers.

Suppose $f'(x_k) \neq 0$, then Newton's method iterates

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

In specific condition, x_k convergence to a root of $f(x) = 0$ fast as $k \to \infty$.

Outline

Antiderivatives/Indefinite Integral

1. Any function F satisfying $F' = f$ is called an *antiderivative* (or a primitive function) of f.
2. Obviously, if F is an antiderivative of f, then so is $F + C$ for any constant C, since $\dfrac{dC}{dx} = 0$.
3. Note that if F and G are two antiderivatives of f on an open interval, then we have $G(x) - F(x) = C$ for some constant C. The *indefinite integral* notation
$$\int f(x)dx$$
is nothing but a new dress of the antiderivatives! The function $f(x)$ appearing in an indefinite integral is usually called the *integrand*.
4. For constants a and b, we have
$$\int (af(x) + bg(x))dx = a\int f(x)dx + b\int g(x)dx$$

Indefinite Integral

$$\frac{d}{dx}\frac{1}{p+1}x^{p+1} = x^p \quad \overset{p\neq -1}{\Longleftrightarrow} \quad \int x^p dx = \frac{1}{p+1}x^{p+1} + C$$

$$\frac{d}{dx}e^x = e^x \quad \Longleftrightarrow \quad \int e^x dx = e^x + C$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \quad \Longleftrightarrow \quad \int \frac{1}{x}dx = \ln|x| + C$$

$$\frac{d}{dx}\sin x = \cos x \quad \Longleftrightarrow \quad \int \cos x dx = \sin x + C$$

$$\frac{d}{dx}[-\cos x] = \sin x \quad \Longleftrightarrow \quad \int \sin x dx = -\cos x + C$$

$$\frac{d}{dx}\tan x = \sec^2 x \quad \Longleftrightarrow \quad \int \sec^2 x dx = \tan x + C$$

$$\vdots$$

Initial Value Problems

The constant C appearing in

$$\int f(x)dx = F(x) + C$$

may be determined uniquely if further condition is imposed on the value of the antiderivative at a specific x_0.

Such a value of the antiderivative is usually called an initial value.

Riemann Sums

The process in computing area in above example can obviously be applied to any continuous function f on the interval $[a, b]$.

The so called Riemann sum of a continuous function $f(x)$ on an interval $[a, b]$ with respect to a subdivision of the interval into n subintervals by the points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

is a straightforward generalization of rectangular approximation of area, which is defined by:

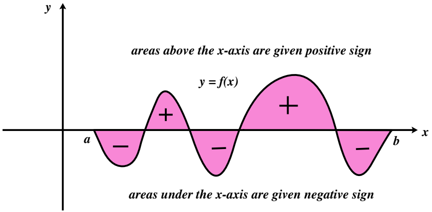
$$S_n = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

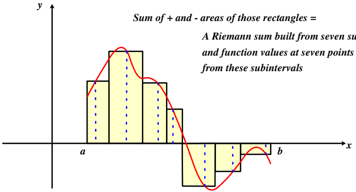
1. If $c_i = x_{i-1}$ for all i, then S_n is called a left (left point) Riemann sum.
2. If $c_i = x_i$ for all i, then S_n is called a right Riemann (right point) sum.
3. If $c_i = (x_{i-1} + x_i)/2$ for all i, then S_n is called a middle (middle point) Riemann sum.

Riemann Sums and Signed Area

A Riemann sum is just a rectangular approximation of the **signed area** (+ve/-ve area) between the graph and the x-axis, based on the chosen points x_i's and c_i's.

Just recall that the "rectangular areas" in the Riemann sum could actually mean certain quantity other than area, e.g., displacement.





Riemann Sums and Integrability

The definite integral of a continuous function $f(x)$ on an interval $[a, b]$ can be defined by using subintervals of equal length

$$\Delta x = \frac{b-a}{n};$$

i.e., with subdivision points $a = x_0 < x_1 < x_2 < \cdots < x_i < \cdots x_n = b$, where $x_i = x_0 + i\Delta x$, and c_i in $[x_{i-1}, x_i]$.

If the limit of Riemann sum

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \cdot \Delta$$

exists on real numbers, we say the function f is Riemann integrable if the limit of the Riemann sum exists and has a unique limit L. The limit is called the definite integral of f from a to b, denoted by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \cdot \Delta = L$$

Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)

Let f be a continuous function on the closed interval $[a, b]$. If $F(x)$ is an antiderivative of f, i.e., $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a),$$

which is often denoted as $F(x)\Big|_a^b$ or $\Big[F(x)\Big]_a^b$.

In other words, whenever you can find

$$\int f(x)dx = F(x) + C,$$

it is just one step further to find the corresponding definite integral:

$$\int_a^b f(x)dx = F(b) - F(a).$$

Some Properties of Integrable Functions

Let f and g are integrable on closed interval $[a, b]$, then

1. for any constants A, B, we have

$$\int_a^b [Af(x) + Bg(x)]dx = A\int_a^b f(x)dx + B\int_a^b f(x)dx$$

2. for any constants $a \leq b \leq c$, we have

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

3. if $f(x) \geq g(x)$ on $[a, b]$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$

4. if $a > b$, we define $\int_b^a f(x)dx = -\int_a^b f(x)dx$ in conventional.

Integral Sandwiches for $\sin x$ and $\cos x$

Repeating such procedures, we have (show that by induction)

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots - \frac{x^{4n-1}}{(4n-1)!} \le \sin x \le x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{x^{4n+1}}{(4n+1)!}$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots - \frac{x^{2n-2}}{(2n-2)!} \le \cos x \le 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{x^{2n+2}}{(2n+2)!}$$

We can approximate $\sin x$ and $\cos x$ by polynomial functions

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{x^{4n+1}}{(4n+1)!}$$

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{x^{2n+2}}{(2n+2)!}$$

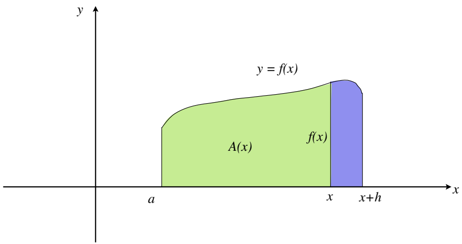
Fundamental Theorem of Calculus (v2)

We consider the following "area function" defined by $A(x) = \int_a^x f(t)dt$. Then $A'(x) = f(x)$ and $A(x)$ is an antiderivative of $f(x)$. We have

Theorem (Fundamental Theorem of Calculus v2)

Let f be a continuous function on the interval $[a, b]$. Then

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$



Net Change Theorem

Just by rewriting the fundamental theorem of calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F'(x) = f(x)$ into another form, we have net change theorem

$$\int_a^b F'(x)dx = F(b) - F(a)$$

since $F(b) - F(a)$ is the change in $y = F(x)$ when x changes from a to b.

The Substitution Rule

Theorem (The Substitution Rule in Indefinite Integral)

If $u = g(x)$ is a differentiable function whose range is an interval I, and $f(x)$ is continuous on I, then (since $u = g(x)$ means $du = g'(x)dx$)

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Theorem (The Substitution Rule in Definite Integral)

If $u = g(x)$ is a differentiable function whose range is an interval I, and $f(x)$ is continuous on I, then

$$\int_a^b f(g(x))g'(x)dx \overset{u=g(x)}{=} \int_{g(a)}^{g(b)} f(u)du.$$