Calculus IB: Detailed Solutions for Multiple Choice (Past Midterm Exam)

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Multiple Choice in Past Midterm Exam

This slides give detailed solutions for multiple choice in past midterm exam. The detailed solutions for other problems in past midterm/final exam can be found on our website.

Problem 1 is choosing the version of the paper, which is not interesting.

Problem 2 (L'Hôpital's rule)

Evaluate $\lim_{x \to 2} \frac{x^2 + 2x - 8}{x^2 - 2x}$

Solution

Note that we have

$$\lim_{x \to 2}(x^2 + 2x - 8) = 0 \quad \textit{and} \quad \lim_{x \to 2}(x^2 - 2x) = 0.$$

Hence, the limit has the form $\frac{0}{0}$ *and we can use L'Hôpital's rule as follows*

$$\lim_{x \to 2} \frac{x^2 + 2x - 8}{x^2 - 2x} = \lim_{x \to 2} \frac{(x^2 + 2x - 8)'}{(x^2 - 2x)'} = \lim_{x \to 2} \frac{2x + 2}{2x - 2} = \frac{2 \cdot 2 + 2}{2 \cdot 2 - 2} = 3$$

Problem 3 (conjugate trick)

Evaluate $\lim_{x\to\infty}(\sqrt{x^2+3x+1}-\sqrt{x^2+x+2})$

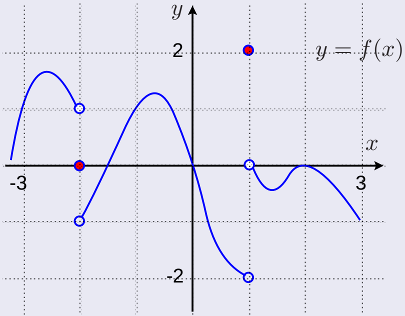
Solution

The form $\infty - \infty$ is undefined and we should use the conjugate trick as follows

$$\begin{aligned}
&\lim_{x\to\infty}(\sqrt{x^2+3x+1}-\sqrt{x^2+x+2}) \\
=&\lim_{x\to\infty}\frac{(\sqrt{x^2+3x+1}-\sqrt{x^2+x+2})(\sqrt{x^2+3x+1}+\sqrt{x^2+x+2})}{\sqrt{x^2+3x+1}+\sqrt{x^2+x+2}} \\
=&\lim_{x\to\infty}\frac{(x^2+3x+1)-(x^2+x+2)}{\sqrt{x^2+3x+1}+\sqrt{x^2+x+2}}=\lim_{x\to\infty}\frac{2x-1}{\sqrt{x^2+3x+1}+\sqrt{x^2+x+2}} \\
=&\lim_{x\to\infty}\frac{\frac{1}{x}\cdot(2x-1)}{\frac{1}{x}\cdot(\sqrt{x^2+3x+1}+\sqrt{x^2+x+2})} \\
=&\frac{\lim_{x\to\infty}(2-\frac{1}{x})}{\lim_{x\to\infty}\left(\sqrt{1+\frac{3}{x}+\frac{1}{x^2}}+\sqrt{1+\frac{1}{x}+\frac{2}{x^2}}\right)}=\frac{2}{1+1}=1
\end{aligned}$$

Problem 4 (understanding one-side limits)

Find $\lim_{x \to -2^+} f(f(x))$ where the function f is give as follows.



Solution

Observe the graph of the function. When $x \to -2^+$*, we have* $u = f(x) \to -1^+$*. Hence*

$$\lim_{x \to -2^+} f(f(x)) = \lim_{u \to -1^+} f(u) = 1$$

Problem 5 (continuity of the function)

A function g is defined by

$$g(x) = \begin{cases} 4x - a + 3 & \text{if } x < 1 \\ ax^2 + 3x & \text{if } x \geq 1 \end{cases}$$

where a is a constant. Find a such that g is continuous for all x.

Solution

Since the function $g_1(x) = 4x - a + 3$ is continuous with domain $x < 1$ and $g_2(x) = ax^2 + 3x$ is continuous with domain $x \geq 1$. We only needs to find a such that $g(x)$ is continuous at $x = 1$, that is

$$\begin{aligned} g(1) = \lim_{x \to 1^+} g(x) \Longrightarrow & g_1(1) = g_2(1) \\ \Longrightarrow & 4 - a + 3 = a + 3 \\ \Longrightarrow & a = 2 \end{aligned}$$

Problem 6 (derivative and slope of the tangent line)

Find the slope of the tangent line to the graph of $y = \frac{4}{x^2}$ at the point $(2, 1)$.

Solution

Since the slope of the tangent line at $(2, 1)$ *is the derivative if the function at* $x = 2$*, we just need to find the derivative and it is unnecessary to plot the graph.*

$$\left.\frac{dy}{dx}\right|_{x=2} = \left.\frac{d\left(\frac{4}{x^2}\right)}{dx}\right|_{x=2} = \left.\left(-\frac{8}{x^3}\right)\right|_{x=2} = -1$$

Problem 7 (quotient rule, product rule)

Find the derivative $f'(2\pi)$ where $f(x) = \dfrac{x \sin x}{\cos x + 1}$.

Solution

Using the quotient rule and product rule to find $f'(x)$

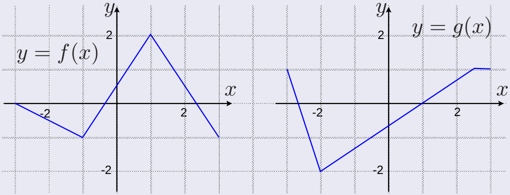
$$\begin{aligned} f'(x) &= \frac{(x \sin x)' \cdot (\cos x + 1) - (x \sin x) \cdot (\cos x + 1)'}{(\cos x + 1)^2} \\ &= \frac{(x' \sin x + x(\sin x)') \cdot (\cos x + 1) - (x \sin x) \cdot (-\sin x)}{(\cos x + 1)^2} \\ &= \frac{(\sin x + x \cos x) \cdot (\cos x + 1) + x \sin^2 x}{(\cos x + 1)^2} \end{aligned}$$

Since $\sin 2\pi = 0$ *and* $\cos 2\pi = 1$, *we have*

$$f'(2\pi) = \frac{(\sin 2\pi + 2\pi \cos 2\pi) \cdot (\cos 2\pi + 1) + 2\pi \sin^2 2\pi}{(\cos 2\pi + 1)^2} = \frac{2\pi \cdot 2 + 2\pi \cdot 0}{2^2} = \pi$$

Problem 8 (chain rule, slope of the line)

Find the derivative $(f \circ g)'(1)$, where the graphs of the functions f and g are given respectively as follows.



Solution

The chain rule means $(f \circ g)'(1) = f'(g(1)) \cdot g'(1)$. *The graph means* $g(1) = 0$ *and* $g'(1)$ *is the the slope of line passes point* $(-2, -2)$ *and* $(1, 0)$ *which means* $g'(1) = \dfrac{-2 - 0}{-2 - 1} = \dfrac{2}{3}$. *Similarly,* $f'(g(1)) = f'(0)$ *is the the slope of line passes point* $(-1, -1)$ *and* $(1, 2)$, *then* $f'(0) = \dfrac{-1 - 2}{-1 - 1} = \dfrac{3}{2}$. *Finally, we have* $(f \circ g)'(1) = \dfrac{2}{3} \cdot \dfrac{3}{2} = 1$.

Problem 9 (derivative of inverse function)

$y = f(x) = x^3 + 1$ is a one to one function. If $y = h(x)$ is the inverse function of f, find $h'(2)$.

Solution

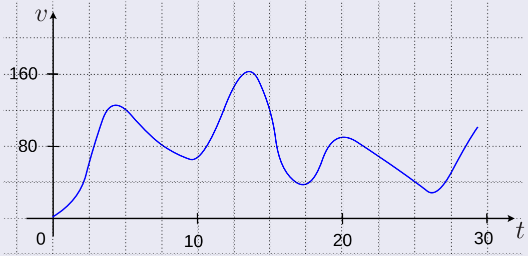
One Solution: The definition $y = x^3 + 1$ *means* $x = (y-1)^{\frac{1}{3}}$, *then* $h(x) = (x-1)^{\frac{1}{3}}$ *and* $h'(x) = \frac{1}{3}(x-1)^{-\frac{2}{3}}$. *We have* $h'(2) = \frac{1}{3}(2-1)^{-\frac{2}{3}} = \frac{1}{3}$.

Another Solution: Since h *is the inverse function of* f *and* $f(1) = 2$, *we have*

$$h'(2) = (f^{-1})'(1) = \frac{1}{f'(1)} = \frac{1}{\frac{d(x^3+1)}{dx}\Big|_{x=1}} = \frac{1}{(3x^2)\Big|_{x=1}} = \frac{1}{3}$$

Problem 10 (application of derivative)

The graph of the velocity function $v = v(t)$ of a runner, in meters per minute, is shown below. At which of the following times is the runner slowing down most rapidly than the other given moments?



(a) $t = 5$ (b) $t = 10$ (c) $t = 15$ (d) $t = 20$ (e) $t = 25$

Solution

Just sketch the tangent lines and compare their slopes. The slope of tangent line at $t = 10$ is positive, at $t = 20$ is closed to 0, at $t = 5$, $t = 15$ and $t = 25$ are negative. At $t = 15$, the slope is negative and smaller than others. Hence the answer is (c) $t = 15$.

Problem 11 (change of rate)

The volume V of a sphere is an increasing function of its area A. What is the rate of change of V with respect to A when $A = 36\text{cm}^2$?

Solution

Let the radius of sphere is r cm, then we have

$$A = 4\pi r^2 \quad \textit{and} \quad r = \sqrt{\frac{A}{4\pi}} \quad (\textit{r should be positive}).$$

Then we have

$$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi \cdot \left(\frac{A}{4\pi}\right)^{\frac{3}{2}} = \frac{A^{\frac{3}{2}}}{6\sqrt{\pi}}$$

and

$$\frac{dV}{dA}\Big|_{A=36} = \frac{d\left(\frac{A^{\frac{3}{2}}}{6\sqrt{\pi}}\right)}{dA}\Bigg|_{A=36} = \frac{3}{2}\cdot\frac{A^{\frac{1}{2}}}{6\sqrt{\pi}}\Bigg|_{A=36} = \frac{3}{2}\cdot\frac{6}{6\sqrt{\pi}} = \frac{3}{2\sqrt{\pi}}.$$