Calculus IB: Lecture 22

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Outline

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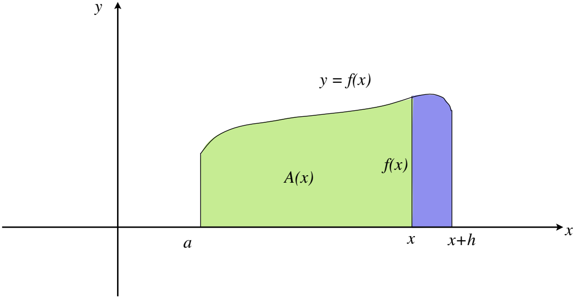
Fundamental Theorem of Calculus (v2)

In the geometric view of fundamental theorem of calculus, we consider the following "area function" defined by

$$A(x) = \int_a^x f(t)dt.$$

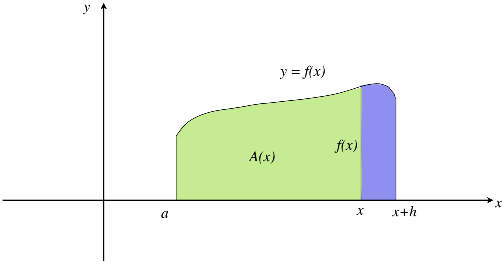


Fundamental Theorem of Calculus (v2)

In the geometric view of fundamental theorem of calculus, we consider the following "area function" defined by

$$A(x) = \int_a^x f(t)dt.$$

Then we study the derivative of $A(x)$.



Fundamental Theorem of Calculus (v2)

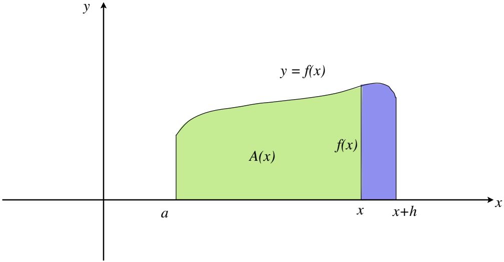
The "area function" is

$$A(x) = \int_a^x f(t)dt.$$

Note that given $h \approx 0$, then

$$A(x+h) - A(x) = \int_x^{x+h} f(t)dt \approx f(x)h$$

and hence we may expect: $A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = f(x)$



Fundamental Theorem of Calculus (v2)

To be precise, just consider the "area sandwich" for $h > 0$:

$$\min_{x \le t \le x+h} f(t)h \le A(x+h) - A(x) \le \max_{x \le t \le x+h} f(t)h$$

As f is continuous on $[a, b]$, we have by taking limits

$$\begin{aligned} f(x) &= \lim_{h \to 0^+} \min_{x \le t \le x+h} f(t) \\ &\le \lim_{h \to 0^+} \frac{A(x+h) - A(x)}{h} \\ &\le \lim_{h \to 0^+} \max_{x \le t \le x+h} f(t) = f(x) \end{aligned}$$

For $h < 0$, we consider the interval $[x + h, x]$ and end up with

$$\lim_{h \to 0^-} \frac{A(x+h) - A(x)}{h} = f(x)$$

Fundamental Theorem of Calculus (v2)

In other word, $A'(x) = f(x)$ and hence the area function $A(x)$ is an antiderivative of $f(x)$. Rewrite this as a theorem, we have:

Theorem (Fundamental Theorem of Calculus v2)

Let f be a continuous function on the interval $[a, b]$. Then

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Fundamental Theorem of Calculus (v2)

Recall that the antiderivative of f can be expressed as

$$\int f(x)dx = F(x) + C,$$

and there must be a constant C such that (fundamental theorem of calculus v2)

$$\int_a^x f(t)dt = F(x) + C$$

Putting in $x = a$, we have

$$0 = \int_a^a f(t)dt = F(a) + C \quad \iff \quad C = -F(a)$$

Therefore we have

$$\int_a^x f(t)dt = F(x) - F(a)$$

which corresponds to previous version of fundamental theorem of calculus

$$\int_a^b f(t)dt = F(b) - F(a)$$

Fundamental Theorem of Calculus (v2)

Example

Let $y = \int_1^{x^2} \sin 3t dt$, find $\dfrac{dy}{dx}$

Let $u = x^2$ and use chain rule, then

$$\frac{dy}{dx} = \left(\frac{d}{du}\int_1^{u} \sin 3t dt\right) \cdot \frac{du}{dx}$$
$$= (\sin 3u) \cdot \frac{du}{dx} = (\sin 3x^2) \cdot \frac{dx^2}{dx} = 2x \sin 3x^2$$

Exercise

Instead of using the fundamental theorem of calculus directly, try to use the limit definition of derivative to find derivatives above.

Outline

Net Change Theorem

Just by rewriting the fundamental theorem of calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F'(x) = f(x)$ into another form, we have net change theorem

$$\int_a^b F'(x)dx = F(b) - F(a)$$

since $F(b) - F(a)$ is the change in $y = F(x)$ when x changes from a to b.

Net Change Theorem

Example

A particle moves along a line with velocity function $v(t) = t^2 - 2t$ (meters per second). Find the displacement and distance traveled during the time interval $1 \leq t \leq 6$.

The displacement is

$$s(6) - s(1) = \int_1^6 (t^2 - 2t)dt = \left[t^3/3 - t^2\right]_1^6 = (72 - 36) - (1/3 - 1) = \frac{110}{3}$$

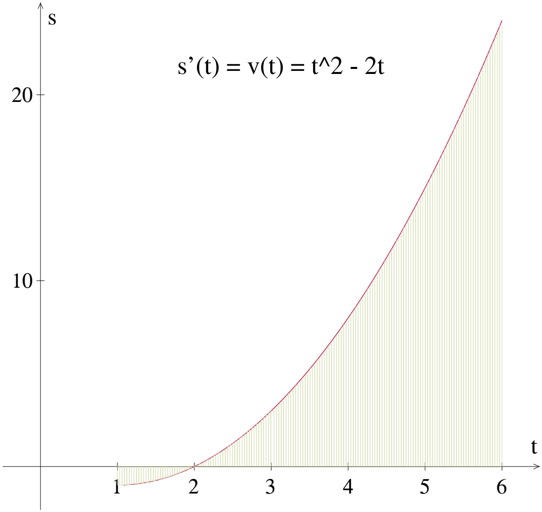
Note that the speed function is $|v(t)| = |t^2 - 2t|$. Hence the distance traveled is

$$\int_1^6 |t(t-2)|dt = \int_1^2 -(t^2 - 2t)dt + \int_2^6 (t^2 - 2t)dt$$
$$= \left[-t^3/3 + t^2\right]_1^2 + \left[t^3/3 - t^2\right]_2^6 = 38,$$

where 2 is the point the sign of $t(t-2)$ changes.

Net Change Theorem

Geometric view of above example:



Example

Suppose a volcano spewed out solid materials is of the rate $r(t)$ at which are given in the following table.

t(in seconds)	0	1	2	3	4	5	6
$r(t)$(tones per second)	2	10	24	36	46	54	60

(a) Give upper and lower estimates for the total quantity $Q(6)$ of erupted materials after 6 seconds.

(b) Use the Midpoint Rule (Midpoint Riemann Sum) to estimate $Q(6)$.

(a) Using the table,

$$2 + 10 + 24 + 36 + 46 + 54 < \int_0^6 r(t)dt < 10 + 24 + 36 + 46 + 54 + 60$$

$$172 < Q(6) < 230 \text{ (tones)}$$

(b) Using three subintervals of length 2, with subdivision points $0 < 2 < 4 < 6$, we have midpoints $1, 3, 5$ and hence

$$Q(6) \approx 2(10 + 36 + 54) = 200 \text{ (tones)}$$

Outline

The Substitution Rule

Theorem (The Substitution Rule in Indefinite Integral)

If $u = g(x)$ is a differentiable function whose range is an interval I, and $f(x)$ is continuous on I, then (since $u = g(x)$ means $du = g'(x)dx$)

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Theorem (The Substitution Rule in Definite Integral)

If $u = g(x)$ is a differentiable function whose range is an interval I, and $f(x)$ is continuous on I, then

$$\int_a^b f(g(x))g'(x)dx \overset{u=g(x)}{=} \int_{g(a)}^{g(b)} f(u)du.$$

The Substitution Rule

Example

Find $\int_0^2 \sqrt{4x+1}dx$

Let $u = g(x) = 4x+1$, then $g(0) = 1$, $g(2) = 9$, $\sqrt{4x+1} = u^{\frac{1}{2}}$ and

$$\int_0^2 \sqrt{4x+1}dx \overset{u=4x+1}{=} \int_1^9 \frac{1}{4}u^{\frac{1}{2}}du = \frac{1}{4}\left[\frac{2}{3}u^{3/2}\right]_1^9 = \frac{1}{6}(27-1) = \frac{13}{3}$$

Example

Find $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x dx$

We hope the expression only depends on $\cos x$ or $\sin x$:

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x dx = \int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \ (\sin x \ dx)$$

$$= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \cos^4 x \ (-d \cos x)$$

Let $u = g(x) = \cos x$, then $g(0) = \cos 0 = 1$, $g(1) = \sin \frac{\pi}{2} = 0$ and

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x dx = -\int_{g(0)}^{g(\frac{\pi}{2})} (1 - u^2) u^4 du$$

$$= -\left[\frac{t^5}{5} - \frac{t^7}{7}\right]_1^0 = -\left[0 - \left(\frac{1}{5} - \frac{1}{7}\right)\right] = \frac{2}{35}$$

The Substitution Rule

In above example, we use the fact

$$\int_{g(0)}^{g(\frac{\pi}{2})} (1-u^2)u^4 du = \int_{1}^{0} (1-u^2)u^4 du$$

where $g(x) = \cos x$.

The range of the definite integration should follow by $g(\frac{\pi}{2}) = 0$ and $g(0) = 1$. The expression allows $1 > 0$.

The Substitution Rule

Theorem (The Substitution Rule in Definite Integral)

If $u = g(x)$ is a differentiable function whose range is an interval I, and $f(x)$ is continuous on I, then

$$\int_a^b f(g(x))g'(x)dx \overset{u=g(x)}{=} \int_{g(a)}^{g(b)} f(u)du.$$

If we want to find

$$\int_{u_1}^{u_2} f(u)du,$$

we can also use the substitution rule by take $u = g(x)$ and compute

$$\int_{x_1}^{x_2} f(g(x))g'(x)dx,$$

where $x_1 = g^{-1}(u_1)$ and $x_2 = g^{-1}(u_2)$ (suppose the inverse function of g exists in the interval).

The Substitution Rule

The expression

$$\int_{x_1}^{x_2} f(g(x))g'(x)dx$$

looks more complicated than

$$\int_{u_1}^{u_2} f(u)du.$$

However, such substitution could be very useful in some specific problem.

The Substitution Rule

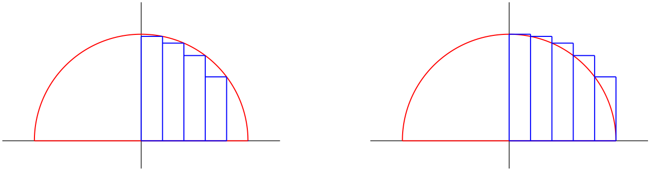
Recall that the area of quarter circle with radius r is

$$\frac{1}{4}\pi r^2 = \int_0^r \sqrt{r^2 - x^2}dx$$

It is difficult to verify this result by Riemann sum:

$$\lim_{n\to\infty} \frac{r}{n}\sum_{k=1}^{n}\sqrt{r^2 - \left(\frac{k}{n}\right)^2 \cdot r^2}$$

However, we can use substitution rule to find this definite integral.



The Substitution Rule

Let $x = r\sin\theta$. Since the function $\sin\theta$ is a one-to-one function in $[0, \frac{\pi}{2}]$ and $r\sin 0 = 0$, $r\sin\frac{\pi}{2} = r$, we have

$$\begin{aligned}
\int_0^r \sqrt{r^2 - x^2}dx &= \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2\theta}\, d(r\sin\theta) \\
&= \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2\theta} \cdot r\cos\theta d\theta \\
&= r^2 \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2\theta} \cos\theta d\theta \\
&= r^2 \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta \\
&= r^2 \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2} d\theta \\
&= r^2 \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}} = r^2\left[\frac{\pi}{4} + \frac{\sin\pi}{4} - \left(0 + \frac{\sin 0}{4}\right)\right] = \frac{1}{4}\pi r^2
\end{aligned}$$

Since the circle is symmetric, the result

$$\frac{1}{4}\pi r^2 = \int_0^r \sqrt{r^2 - x^2}dx$$

means the area of semi-circle is

$$\frac{1}{2}\pi r^2 = \int_{-r}^r \sqrt{r^2 - x^2}dx$$

and the area of the circle is πr^2.

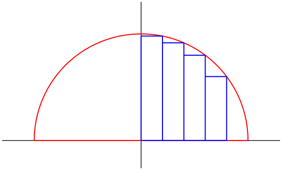
Symmetry in Definite Integrals

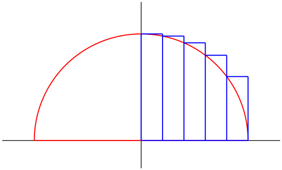
Recall that the integrand $f(x) = r\sqrt{r^2 - x^2}$ in above example is an even function, that is $f(x) = f(-x)$ and the interval of the definite integral

$$\int_{-r}^{r} \sqrt{r^2 - x^2} dx$$

is also symmetric with respect to 0. Hence

$$\int_{-r}^{r} \sqrt{r^2 - x^2} dx = 2\int_{0}^{r} \sqrt{r^2 - x^2} dx = 2\int_{-r}^{0} \sqrt{r^2 - x^2} dx$$





Symmetry in Definite Integrals

Consider a "difficult" problem

$$\int_{-1}^{1} \frac{\tan x}{1 + 3x^2 + 5x^4 + 7x^6} = ?$$

- Applying Riemann sum is very complicated.
- Using fundamental theorem of calculus is also very difficult.

Is this problem real "difficult"?

Symmetry in Definite Integrals

We define

$$f(x) = \frac{\tan x}{1 + 3x^2 + 5x^4 + 7x^6}.$$

In fact, $f(x)$ is an odd function, that is

$$f(x) = -f(-x).$$

Recall that definite integral is the signed area between the graph of the function and the x-axis.

Odd function means the graph is symmetric with respect to origin.

Since the range $[-1, 1]$ also is symmetric and $f(x)$ is well defined on it, we must have

$$\int_{-1}^{1} f(x)dx = 0$$

Symmetry in Definite Integrals

Since the signed areas are canceled, we must have

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} \frac{\tan x}{1+3x^2+5x^4+7x^6}dx = 0.$$