Calculus IB: Lecture 21

Luo Luo

Department of Mathematics, HKUST

http://luoluo.people.ust.hk/

Outline

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Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)

Let f be a continuous function on the closed interval $[a, b]$. If $F(x)$ is an antiderivative of f, i.e., $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a),$$

which is often denoted as $F(x)\big|_a^b$ or $\big[F(x)\big]_a^b$.

In other words, whenever you can find

$$\int f(x)dx = F(x) + C,$$

it is just one step further to find the corresponding definite integral:

$$\int_a^b f(x)dx = F(b) - F(a).$$

The Proof of Fundamental Theorem of Calculus

Partition the interval $[a, b]$ with n subintervals of equal length $\dfrac{b-a}{n}$.

By the mean value theorem Theorem, there exists x^* in $[x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1}) = f(x_i^*) \cdot \frac{b-a}{n}$$

Then, we take the Riemann sum of the function f on $[a, b]$:

$$\sum_{i=1}^{n} f(x_i^*) \cdot \frac{b-a}{n} = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(x_n) - F(x_0) = F(b) - F(a).$$

Thus by the definition of definite integral, we have

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \frac{b-a}{n} = F(b) - F(a).$$

Example of Fundamental Theorem of Calculus

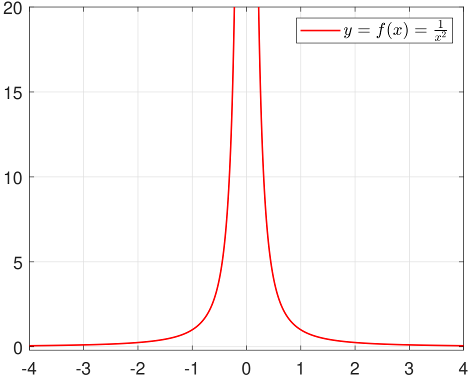
Example

Find the definite integral $\int_{-1}^{2} \frac{1}{x^2} dx$

Since $\left(-\frac{1}{x}\right)' = \frac{1}{x^2}$, we use fundamental theorem of calculus to obatin

$$\int_{-1}^{2} \frac{1}{x^2} dx = \left(-\frac{1}{x}\right)\Bigg|_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}$$

Example of Fundamental Theorem of Calculus



The definite integral $\int_{-1}^{2} \frac{1}{x^2} dx$ can NOT be negative!!!

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Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)

Let f be a continuous function on the closed interval $[a, b]$. If $F(x)$ is an antiderivative of f, i.e., $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a),$$

which is often denoted as $F(x)\big|_a^b$ or $\big[F(x)\big]_a^b$.

The function $f(x) = \frac{1}{x^2}$ is NOT continuous at 0 and $\int_{-1}^{2} \frac{1}{x^2} dx \neq -\frac{3}{2}$.

In fact, if one uses Riemann sum to approximate this integral, the limit of the Riemann sum is ∞.

Riemann Sums and Integrability

The definite integral of a continuous function $f(x)$ on an interval $[a, b]$ can be defined by using subintervals of equal length

$$\Delta x = \frac{b-a}{n};$$

i.e., with subdivision points $a = x_0 < x_1 < x_2 < \cdots < x_i < \cdots x_n = b$, where $x_i = x_0 + i\Delta x$, and c_i in $[x_{i-1}, x_i]$.

If the limit of Riemann sum

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \cdot \Delta$$

exists on real numbers, we say the function f is Riemann integrable if the limit of the Riemann sum exists and has a unique limit L. The limit is called the definite integral of f from a to b, denoted by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \cdot \Delta = L$$

Integrable Functions

Theorem (sufficient condition)

If f is continuous on on $[a, b]$, then f must be (Riemann) integrable.

Theorem (sufficient condition$^+$)

If f is continuous over $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuous points, then f is integrable on $[a, b]$.

Theorem (necessary condition)

If f is (Riemann) integrable on $[a, b]$, then f must be bounded on $[a, b]$.

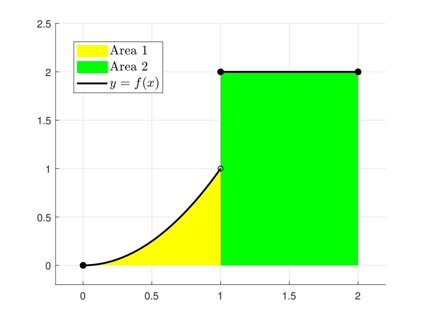
Integrable Functions

Example

The function

$$f(x) = \begin{cases} x^2 & 0 \leq x < 1 \\ 2 & 1 \leq x \leq 2 \end{cases}$$

is integrable om $[0, 2]$.



Integrable Functions

The definite integral of

$$f(x) = \begin{cases} x^2 & 0 \leq x < 1 \\ 2 & 1 \leq x \leq 2 \end{cases}$$

on $[0, 2]$ is

$$\int_0^2 f(x)dx = \lim_{t \to 1^-} \int_0^t f(x)dx + \int_1^2 f(x)dx$$

Non-Integrable Functions

Theorem (necessary condition)

If f is (Riemann) integrable on $[a, b]$, then f must be bounded on $[a, b]$.

Example

The function $f(x) = \dfrac{1}{x^2}$ is unbounded on $[-1, 2]$.

Even if $M > 0$ is sufficient large, we have $x_M = \dfrac{1}{\sqrt{M+1}}$ in $[-1, 2]$

such that $f(x_M) = M + 1 > M$.

Non-Integrable Functions

Theorem (sufficient condition^{+})

If f is continuous over $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuous points, then f is integrable on $[a, b]$.

Example (Dirichlet Function)

Dirichlet function is defined as follows

$$D(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number,} \\ 0 & \text{if } x \text{ is not a rational number.} \end{cases}$$

Dirichlet function is nowhere continuous (in other words, there are infinite number of discontinuous points) and not (Riemann) integrable on any $[a, b]$ when $a < b$.

Non-Integrable Functions

Consider the behavior of Dirichlet function

$$D(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number,} \\ 0 & \text{if } x \text{ is a irrational number.} \end{cases}$$

on interval $[a, b]$, where $a < b$.

Let $a = 0$ and $b = 1$. given rational number 0.123, we can construct infinite irrational numbers

$$0.123x_1x_2x_3 \dots x_n \dots$$

Since each x_i can be select from $0, 1, \dots, 9$, we can think

$$\frac{\#\text{rational numbers}}{\#\text{irrational numbers}} \approx \lim_{n\to\infty} \left(\frac{1}{10}\right)^n = 0$$

Intuitively, rationals number in $[0, 1]$ is much less than irrational number.

Non-Integrable Functions

In other words, $D(x) = 0$ almost everywhere. If we think the area of a segment is 0, then it is reasonable that the "area" of "graph" under $D(x)$ on interval $[a, b]$ is also 0.

In fact, we can define other types of integration (not Riemann integration) to characterize the area under the graph of a function.

Non-Integrable Functions

For example, in the view of Lebesgue integration, Dirichlet function is integrable on $[a, b]$ and

$$\int_a^b D(x)dx = 0.$$

However, the expression

$$\int_a^b D(x)dx$$

is undefined by Riemann integration.

Integrable Functions

In homework and exam of MATH 1013, “integrable” always refers to Riemann integrable.

Some Properties of Integrable Functions

Let f and g are integrable on closed interval $[a, b]$, then

1. for any constants A, B, we have
$$\int_a^b [Af(x) + Bg(x)]dx = A\int_a^b f(x)dx + B\int_a^b f(x)dx$$
2. for any constants $a \leq b \leq c$, we have
$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$
3. if $f(x) \geq g(x)$ on $[a, b]$, then
$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$
4. if $a > b$, we define $\int_b^a f(x)dx = -\int_a^b f(x)dx$ in conventional.

Outline

Integral Sandwiches for $\sin x$ and $\cos x$

Consider that $\cos x \leq 1$ and $\sin x < x$. Then for any $x \geq 0$,

$$\cos x \leq 1 \Longrightarrow \int_0^x \cos t dt \leq \int_0^x 1 dt \Longleftrightarrow \sin t \Big|_0^x \leq t \Big|_0^x$$

$$\sin x \leq x \Longrightarrow \int_0^x \sin t dt \leq \int_0^x t dt \Longleftrightarrow -\cos t \Big|_0^x \leq \frac{x^2}{2}$$

$$1 - \frac{x^2}{2} \leq \cos x \Longrightarrow \int_0^x \left(1 - \frac{t^2}{2}\right) dt \leq \int_0^x \cos t dt = \sin x$$

$$x - \frac{x^3}{3!} \leq \sin x \Longrightarrow \int_0^x \left(t - \frac{t^3}{3!}\right) dt \leq \int_0^x \sin t dt = -\cos x + 1$$

$$\cdots\cdots \Longrightarrow \cdots\cdots$$

Exclamation mark "!" means factorial, that is, $k! = 1 \cdot 2 \cdot \cdots \cdot k$ for any positive integer k. We define $0! = 1$ in conventional.

Integral Sandwiches for $\sin x$ and $\cos x$

Repeating such procedures, we have (show that by induction)

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots - \frac{x^{4n-1}}{(4n-1)!} \leq \sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{x^{4n+1}}{(4n+1)!}$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots - \frac{x^{2n-2}}{(2n-2)!} \leq \cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{x^{2n+2}}{(2n+2)!}$$

We can approximate $\sin x$ and $\cos x$ by polynomial functions

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{x^{4n+1}}{(4n+1)!}$$

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{x^{2n+2}}{(2n+2)!}$$

Integral Sandwiches for $\sin x$ and $\cos x$

We can use the polynomial

$$p(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

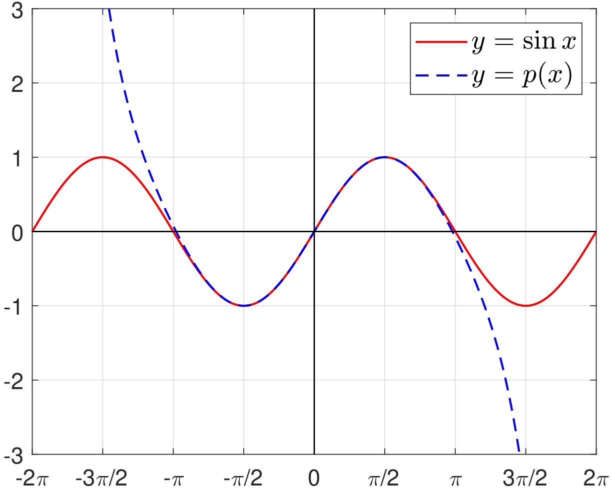
to estimate the function value of $\sin x$, then we have

$$p(x) \leq \sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \Longrightarrow \sin x - p(x) \leq \frac{x^9}{9!}$$

If we restrict x on $\left[0, \frac{\pi}{4}\right]$, the above inequalities implies

$$0 \leq \sin x - p(x) \leq \frac{x^9}{9!} \leq \frac{\left(\frac{\pi}{4}\right)^9}{9!} = 3.13 \times 10^{-7}$$

Integral Sandwiches for $\sin x$ and $\cos x$



We can use the polynomial

$$q(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

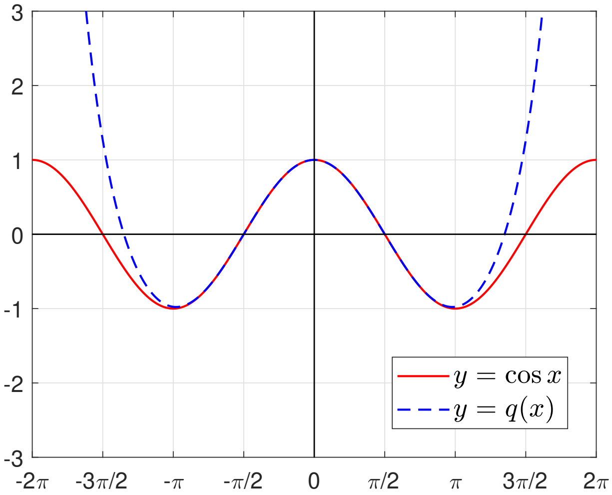
to estimate the function value of $\cos x$, then we have

$$q(x) \le \cos x \le 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \Longrightarrow 0 \le \cos x - q(x) \le \frac{x^8}{8!}$$

If we restrict x on $\left[0, \frac{\pi}{4}\right]$, the above inequalities implies

$$0 \le \cos x - q(x) \le \frac{x^8}{8!} \le \frac{\left(\frac{\pi}{4}\right)^8}{8!} = 3.59 \times 10^{-6}$$

Integral Sandwiches for $\sin x$ and $\cos x$



Taylor Series and Linear Approximation

More general, for $f(x)$ that is infinitely differentiable, we can approximate it by Taylor series

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

In above examples, we take $a = 0$ and $f(x)$ be $\sin x/\cos x$.

We can also think this strategy is an extension of linear approximation

$$f(x) \approx f(a) + f'(a)(x-a)$$

Taylor Series and Optimization

Let $a = x_k$, we have

$$f(x) \approx f(x_k) + f'(x_k)(x - x_k) + \frac{f''(x_k)}{2}(x - x_k)^2 + \frac{f'''(x_k)}{3!}(x - x_k)^3 + \cdots$$

The iteration of gradient descent is

$$\begin{aligned} x_{k+1} &= x_k - \frac{1}{L} f'(x_k) \\ &= \arg\min_x \left[f(x_k) + f'(x_k)(x - x_k) + \frac{L}{2}(x - x_k)^2 \right] \end{aligned}$$

Recall that we suppose $f''(x) \leq L$ for positive L in the analysis of convex optimization, which means the update is optimizing and upper bound of first three terms in Taylor series.

It is easy to check

$$\begin{aligned} x_{k+1} =& x_k - \frac{1}{L} f'(x_k) \\ =& \arg\min_x \left[f(x_k) + f'(x_k)(x - x_k) + \frac{L}{2}(x - x_k)^2. \right] \end{aligned}$$

We define

$$g(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{L}{2}(x - x_k)^2,$$

then $g''(x) = L > 0$ and $g'(x) = f'(x_k) + L(x - x_k)$.

Hence, $g(x)$ is convex and x_{k+1} is its unique critical point (minimizer).

Taylor Series and Optimization

What happens if we directly minimize the first three terms?

$$f(x) \approx f(x_k) + f'(x_k)(x - x_k) + \frac{f''(x_k)}{2}(x - x_k)^2 + \frac{f'''(x_k)}{3!}(x - x_k)^3 + \cdots$$

We define $h(x) = f(x_k) + f'(x_k)(x - x_k) + \dfrac{f''(x)}{2}(x - x_k)^2$, then $h''(x) = f''(x)$ and $h'(x) = f'(x_k) + f''(x_k)(x - x_k)$.

Hence, if $f(x)$ is strictly-convex then $h''(x) = f''(x_k) > 0$ is convex and x_{k+1} is its unique critical point (minimizer) of $h(x)$ is

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)},$$

which leads to Newton's Method!

Taylor Series and Optimization

In theoretical, we can also optimize

$$l(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{f''(x_k)}{2}(x - x_k)^2 + \frac{f'''(x_k)}{3!}(x - x_k)^3$$

to establish an optimization algorithm, but solving such sub-problem is more complicated and difficult to be extended to high-dimensional case.

Taylor Series and Optimization

Similar to linear approximation, the approximation

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

has high accuracy when x is close to a.

Intuitively, if x is far away from a, we require a larger n to increase $n!$ and control the magnitude of

$$\frac{f^{(n)}(a)}{n!}(x-a)^n.$$