Calculus IB: Lecture 20

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Outline

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Riemann Sums and Definite Integrals

The definite integral of a continuous function $f(x)$ on an interval $[a, b]$ can be defined by using subintervals of equal length

$$\Delta x = \frac{b-a}{n};$$

i.e., with subdivision points

$$a = x_0 < x_1 < x_2 < \cdots < x_i < \cdots x_n = b$$

where $x_i = x_0 + i\Delta x$, and c_i in $[x_{i-1}, x_i]$, such that

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\frac{b-a}{n} \qquad \text{whenever the limit exists.}$$

The definite integral exists means the above limit exists on real number and its value does not depend on the choice of c_i.

Riemann Sums and Definite Integrals

In this section, we suppose all of definite integrals exist if we use notation

$$\int_a^b f(x)dx.$$

The existence of definite integral is not required in our course. We will give a brief sketch for this topic in next week.

Example: $\int_0^1 x dx = \frac{1}{2}$

Consider the partition of $[0, 1]$ into n subintervals by the subdivision points

$$0 < \frac{1}{n} < \frac{2}{n} < \frac{3}{n} < \cdots < \frac{n}{n} = 1.$$

We have the left-endpoint Riemann sum is

$$\begin{aligned} & 0 \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n} + \frac{2}{n} \cdot \frac{1}{n} + \cdots + \frac{n-1}{n} \cdot \frac{1}{n} \\ = & \frac{1}{n^2} \cdot [0 + 1 + 2 + 3 + \cdots + (n-1)] \\ = & \frac{1}{n^2} \cdot \frac{(n-1)n}{2} = \frac{1 - \frac{1}{n}}{2} \end{aligned}$$

which tends to the limit $\frac{1}{2}$ as $n \to +\infty$.

Example: $\int_0^1 x dx = \frac{1}{2}$

Similarly, we have the right-endpoint Riemann sum is

$$
\begin{aligned}
&\frac{1}{n} \cdot \frac{1}{n} + \frac{2}{n} \cdot \frac{1}{n} + \cdots + \frac{n}{n} \cdot \frac{1}{n} \\
=&\frac{1}{n^2} \cdot [1 + 2 + 3 + \cdots + n] \\
=&\frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{1 + \frac{1}{n}}{2}
\end{aligned}
$$

which also tends to the limit $\frac{1}{2}$ as $n \to +\infty$.

Example: $\int_0^1 x dx = \frac{1}{2}$

Note that any other Riemann sum

$$S_n = \frac{1}{n}[c_1 + c_2 + \cdots + c_n]$$

with respect to the partition of the interval above, and c_i in $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, is squeezed between the left-endpoint and right-endpoint Riemann sums

$$\frac{1}{2}\left(1 - \frac{1}{n}\right) \leq S_n \leq \frac{1}{2}\left(1 + \frac{1}{n}\right)$$

since the thin rectangular area with height $f(c_i) = c_i$ over the subinterval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ is squeezed by

$$\frac{1}{n} \cdot \frac{i-1}{n} \leq \frac{1}{n} \cdot c_i \leq \frac{1}{n} \cdot \frac{i}{n}.$$

Example: $\int_0^1 xdx = \frac{1}{2}$

Summing over the thin rectangular areas

$$\frac{1}{n} \cdot \frac{i-1}{n} \leq \frac{1}{n} \cdot c_i \leq \frac{1}{n} \cdot \frac{i}{n},$$

We have

$$\lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{n}\right) \leq \lim_{n\to\infty} \frac{1}{n}\left[c_1 + c_2 + \cdots + c_n\right] \leq \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{n}\right),$$

then

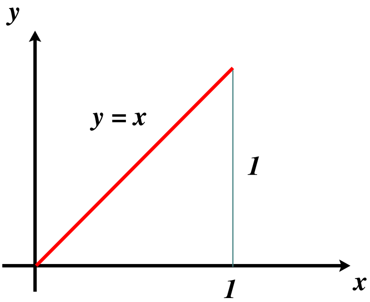
$$\int_0^1 xdx = \lim_{n\to\infty} \frac{1}{n}\left[c_1 + c_2 + \cdots + c_n\right] = \frac{1}{2}.$$

Example: $\int_0^1 x dx = \frac{1}{2}$

If we let x be time and velocity of a partical is be $v = f(x) = x$, then the definite integral (corresponds to the area under the curve)

$$\int_0^1 f(x)dx = \frac{1}{2}$$

is the displacement from time 0 to 1.



Example: $\int_0^1 e^x dx$

Find the value of the definite integral

$$\int_0^1 e^x dx$$

by working with left-endpoint Riemann sums.

We consider the subdivision points are: $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{k}{n}, \ldots, \frac{n}{n} = 1$.

Computing the function values at $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$, the left-endpoint Riemann sum S_n is

$$\begin{aligned} S_n =& \frac{1}{n}\left[e^0 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}}\right] \\ =& \frac{1}{n}\left[1 + e^{\frac{1}{n}} + \left(e^{\frac{1}{n}}\right)^2 + \cdots + \left(e^{\frac{1}{n}}\right)^{n-1}\right] \end{aligned}$$

Example: $\int_0^1 e^x dx$

The left-endpoint Riemann sum S_n is

$$S_n = \frac{1}{n}\left[e^0 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}}\right]$$
$$= \frac{1}{n}\left[1 + e^{\frac{1}{n}} + \left(e^{\frac{1}{n}}\right)^2 + \cdots + \left(e^{\frac{1}{n}}\right)^{n-1}\right]$$

Using the formula $1 + t + t^2 + \cdots + t^{n-1} = \frac{t^n - 1}{t-1}$ for any $t \neq 1$, we have

$$\int_0^1 e^x dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{n}\frac{\left(e^{\frac{1}{n}}\right)^n - 1}{e^{\frac{1}{n}} - 1} = \frac{e-1}{\lim\limits_{n\to\infty} \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}}} = e - 1$$

where we use the result $\lim\limits_{n\to\infty} \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} = 1$.

Then we check the limit $\lim_{n\to\infty} \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} = 1$.

Let $x = \frac{1}{n}$, then $n \to \infty$ means $x \to 0$ and we have

$$\lim_{n\to\infty} \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} = \lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1,$$

where we use L'Hôpital's rule (check the conditions!).

Example: Area of a Circle $= \pi r^2$

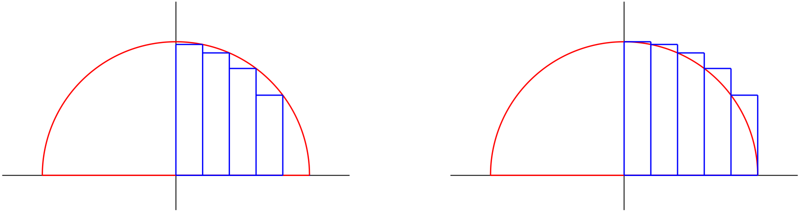
In geometry, the area enclosed by a circle of radius r is πr^2. How to explain this result in the view of Riemann sums/definite integral?

Example: Area of a Circle $= \pi r^2$

Based on the symmetric property of circle, we only needs to consider the right-top part of the circle.

We start by subdividing the interval $[0, 1]$ into n subintervals of the same length $\frac{1}{n}$ by the subdivision points

$$0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n}{n} = 1$$

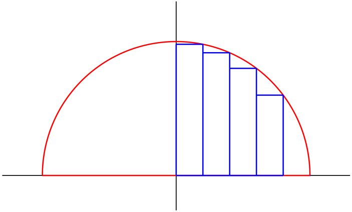


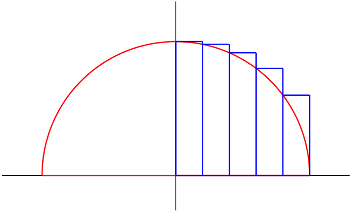
Example: Area of a Circle $= \pi r^2$

The equation of a circle with radius r and center origin is $x^2 + y^2 = r^2$. Hence, we have $y = \sqrt{r^2 - x^2}$.

Over the interval $\left[\frac{(k-1)r}{n}, \frac{kr}{n}\right]$, $k = 1, \ldots, n$, we have the area sandwich:

$$\frac{1}{n} \cdot \sqrt{r^2 - \left(\frac{k}{n}\right)^2 \cdot r^2} < A_k < \frac{1}{n} \cdot \sqrt{r^2 - \left(\frac{k-1}{n}\right)^2 \cdot r^2}$$





Example: Area of a Circle $= \pi r^2$

Following the previous trick, we should compute the limits

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sqrt{r^2 - \left(\frac{k}{n}\right)^2 \cdot r^2}$$

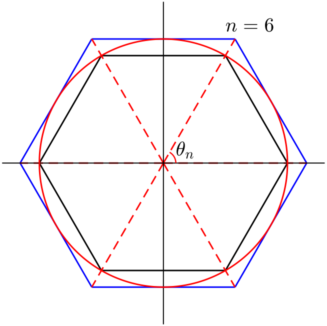
or

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sqrt{r^2 - \left(\frac{k-1}{n}\right)^2 \cdot r^2}$$

How to compute these limits?

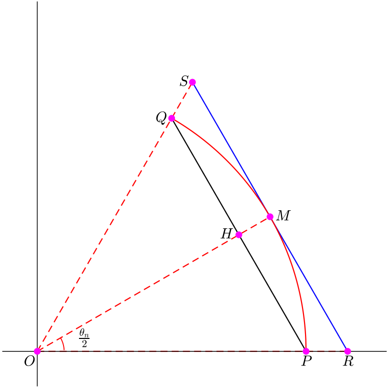
Example: Area of a Circle $= \pi r^2$

We can use triangles to establish the sandwich. Consider the area enclosed by the blue and black polygons when $n \to \infty$ and show that the area of a circle with radius r is πr^2.



Our analysis cannot use the result that the area of circular sector is $\frac{1}{2}r^2\theta_n$, since it is based on the area of the circle is πr^2.

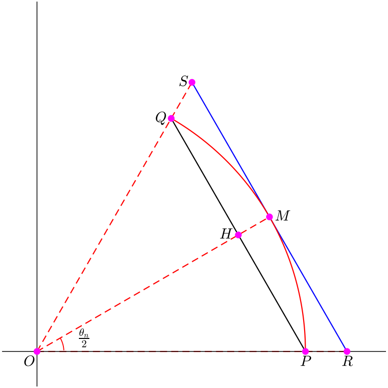
Example: Area of a Circle $= \pi r^2$



$$\text{Area of the circle } = n \cdot \text{Area(Sector } OPMQ)$$

$$\text{Area}(\triangle OPQ) < \text{Area(Sector } OPMQ) < \text{Area}(\triangle ORS)$$

Example: Area of a Circle $= \pi r^2$



$$OP = OM = OQ = r$$

$$OH = r \cos \frac{\theta_n}{2}$$

$$PQ = 2PH = 2r \sin \frac{\theta_n}{2}$$

$$MR = MS = OM \tan \frac{\theta_n}{2} = r \tan \frac{\theta_n}{2}$$

$$RS = 2MR = 2r \tan \frac{\theta_n}{2}$$

$$\text{Area}(\triangle OPQ) = \frac{1}{2} \cdot PQ \cdot OH = r^2 \sin \frac{\theta_n}{2} \cos \frac{\theta_n}{2}$$

$$\text{Area}(\triangle ORS) = \frac{1}{2} \cdot RS \cdot OM = r^2 \tan \frac{\theta_n}{2}$$

Example: Area of a Circle $= \pi r^2$

Let $A_k = \text{Area}(\text{Sector } OPMQ)$, then the area of the circle is

$$S = \sum_{k=1}^{n} A_k = A_1 + A_2, \ldots, A_n.$$

Hence, we have

$$r^2 \sin\frac{\theta_n}{2}\cos\frac{\theta_n}{2} < A_k < r^2 \tan\frac{\theta_n}{2}$$

and

$$n \cdot r^2 \sin\frac{\theta_n}{2}\cos\frac{\theta_n}{2} < S < n \cdot r^2 \tan\frac{\theta_n}{2}.$$

Taking $n \to \infty$, we can obtain $S = \frac{1}{2}\pi r^2$ by sandwich theorem.

Example: Area of a Circle $= \pi r^2$

Let $x = \frac{\pi}{n} \to 0^+$. Since $\theta_n = \frac{2\pi}{n}$, we have

$$\begin{aligned}
&\lim_{n\to\infty} n \cdot r^2 \cdot \sin\frac{\theta_n}{2} \cdot \cos\frac{\theta_n}{2} \\
=&\lim_{n\to\infty} n \cdot r^2 \cdot \sin\frac{\pi}{n} \cdot \cos\frac{\pi}{n} \\
=&\lim_{n\to\infty} \pi r^2 \cdot \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} \cdot \cos\frac{\pi}{n} \\
=&\pi r^2 \cdot \lim_{n\to\infty} \cos\frac{\pi}{n} \cdot \lim_{n\to\infty} \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} \\
=&\pi r^2 \cdot \lim_{x\to 0^+} \cos x \cdot \lim_{x\to 0} \frac{\sin x}{x} \\
=&\pi r^2
\end{aligned}$$

Example: Area of a Circle $= \pi r^2$

Since $x = \dfrac{\pi}{n} \to 0^+$ and $\theta_n = \dfrac{2\pi}{n}$, we have

$$
\begin{aligned}
&\lim_{n\to\infty} n \cdot r^2 \cdot \tan\frac{\theta_n}{2} \\
=&\lim_{n\to\infty} n \cdot r^2 \cdot \frac{\sin\frac{\pi}{n}}{\cos\frac{\pi}{n}} \\
=&\lim_{x\to 0} r^2 \cdot \frac{\pi}{x} \cdot \frac{\sin x}{\cos x} \\
=&\pi r^2 \cdot \lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} \cos x \\
=&\pi r^2
\end{aligned}
$$

Hence, the sandwich theorem means $S = \pi r^2$.

Example: Area of a Circle $= \pi r^2$

Note that our analysis use the identity

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \tag{1}$$

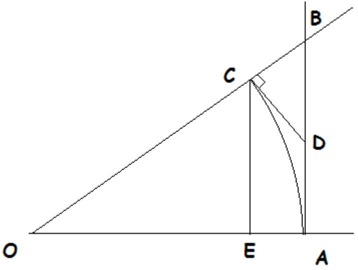
The proof of (1) in Lecture 07 uses

$$\text{Area(circular sector)} = \frac{1}{2} r^2 \theta \tag{2}$$

to show $\sin\theta < \theta < \tan\theta$ when $0 < \theta < \frac{\pi}{2}$.

However, the formula (2) comes from Area(circle) $= \pi r^2$, which leads to circular argument!!! We desire another proof of $\sin\theta < \theta < \tan\theta$ without using the identity of area.

Show $\sin\theta < \theta$ without Area

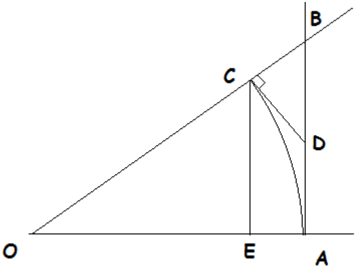


Let $\theta = \angle AOC$ be the corresponding angle of the sector and the lengths of OC and OA is the radius r. Then the length of arc AC is $r\theta$.

The shortest distance from point C to line AO is $CE = r\sin\theta$, where CE is orthogonal to OA. Another path from point C to line OA is arc CA, which is longer than CE because it is not the shortest path. So we have

$$r\sin\theta < r\theta \Longrightarrow \sin\theta < \theta.$$

Show $\theta < \tan\theta$ without Area



Since the set of points bound by sector OCA is a subset of the set of points bound by quadrilateral $OCDA$, the perimeter of quadrilateral $OCDA$ must be longer than the perimeter of the sector OCA (we use both of them are convex sets). Hence, we have $CD + DA >$ arc $CA = r\theta$ and

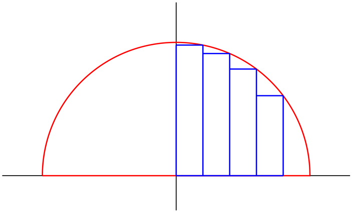
$$r\tan\theta = BA = BD + DA > +CD + DA > r\theta \Longrightarrow \theta < \tan\theta.$$

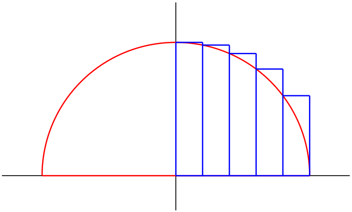
Example: Area of a Circle $= \pi r^2$

The equation of a circle with radius r and center origin is $x^2 + y^2 = r^2$. Hence, the curve of upper semicircle corresponds to $y = \sqrt{r^2 - x^2}$ and we have

$$\text{area of quarter circle} = \int_0^r \sqrt{r^2 - x^2}dx = \frac{\pi r^2}{4}$$

$$\text{area of semicircle} = \int_{-r}^r \sqrt{r^2 - x^2}dx = \frac{\pi r^2}{2}$$





Outline

Calculating definite integrals by the original limit definition based on Riemann sums is very difficult in general.

We have seen that it is so complicated even if we only want to derive the area formula of a circle.

Sometimes, there is an easier way compute definite integrals in general.

Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)

Let f be a continuous function on the closed interval $[a, b]$. If $F(x)$ is an antiderivative of f, i.e., $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a),$$

which is often denoted as $F(x)\Big|_a^b$ or $\Big[F(x)\Big]_a^b$.

In other words, whenever you can find

$$\int f(x)dx = F(x) + C,$$

it is just one step further to find the corresponding definite integral:

$$\int_a^b f(x)dx = F(b) - F(a).$$

Fundamental Theorem of Calculus

Example

Evaluate $\int_0^1 x dx$.

Since $\dfrac{d}{dx}\left(\dfrac{x^2}{2}\right) = x$, we have by the fundamental theorem of calculus

$$\int_0^1 x dx = \frac{1}{2}x^2 \Big|_0^1 = \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 = \frac{1}{2}.$$

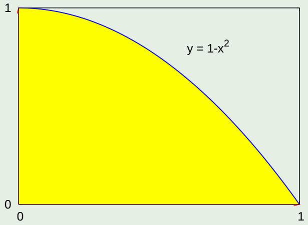
Fundamental Theorem of Calculus

Example

Evaluate $\int_0^1 (1 - x^2)dx$.

Using the fundamental theorem of calculus, we have

$$\int_0^1 (1 - x^2)dx = \left[x - \frac{1}{3}x^3\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$



Example

Find the area under the graph of the function over the given interval:

1. $f(x) = x^2$, $1 \leq x \leq 2$.

$$\int_1^2 x^2 dx = \left[\frac{1}{3}x^3\right]_1^2 = \frac{1}{3}(2)^3 - \frac{1}{3}(1)^3 = \frac{7}{3}$$

2. $g(x) = e^x$, $0 \leq x \leq 3$.

$$\int_0^3 e^x dx = \left[e^x\right]_0^3 = e^3 - e^0 = e^3 - 1$$

Fundamental Theorem of Calculus

Example

Find the area under the graph of the function over the given interval:

1. $h(x) = \sin x,\ 0 \le x \le \pi.$

$$\int_0^{\pi} \sin x dx = \Big[-\cos x\Big]_0^{\pi} = [-\cos(\pi)] - [-\cos 0] = 2$$

2. $u(x) = \cos x,\ 0 \le x \le \pi/2.$

$$\int_0^{\pi/2} \cos x dx = \Big[\sin x\Big]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1$$

Fundamental Theorem of Calculus

Example

A few more definite integrals:

- $$\int_1^2 (5x^4 - 6x^2 + x - 1)dx = \left[x^5 - 2x^3 + \frac{x^2}{2} - x\right]_1^2$$
  $$= [32 - 16 + 2 - 2] - [1 - 2 + \frac{1}{2} - 1] = \frac{35}{2}$$
- $$\int_0^2 (5e^x - 3x^2)dx = \left[5e^x - x^3\right]_0^3 = [5e^3 - 27] - [5e^0 - 0] = 5e^3 - 32$$
- $$\int_1^5 \frac{1}{x}dx = \left[\ln x\right]_1^5 = \ln 5 - \ln 1 = \ln 5$$