Calculus IB: Lecture 19

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Outline

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Indefinite Integral

$$\frac{d}{dx}\frac{1}{p+1}x^{p+1} = x^p \quad \overset{p\neq -1}{\iff} \quad \int x^p dx = \frac{1}{p+1}x^{p+1} + C$$

$$\frac{d}{dx}e^x = e^x \quad \iff \quad \int e^x dx = e^x + C$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \quad \iff \quad \int \frac{1}{x}dx = \ln|x| + C$$

$$\frac{d}{dx}\sin x = \cos x \quad \iff \quad \int \cos x dx = \sin x + C$$

$$\frac{d}{dx}[-\cos x] = \sin x \quad \iff \quad \int \sin x dx = -\cos x + C$$

$$\frac{d}{dx}\tan x = \sec^2 x \quad \iff \quad \int \sec^2 x dx = \tan x + C$$

$$\vdots$$

Initial Value Problems

The constant C appearing in

$$\int f(x)dx = F(x) + C$$

may be determined uniquely if further condition is imposed on the value of the antiderivative at a specific x_0.

Such a value of the antiderivative is usually called an initial value.

No simple result can summarize which types of further condition could lead to uniqueness. This topic is contained in other course, e.g "Ordinary Differential Equations".

Initial Value Problems

Example

Suppose that the graph of $y = y(x)$ defines a curve passing the point $(1, -2)$, with its slope satisfying $y' = x^2$. Find the function y.

We first find the indefinite integral

$$y = \int x^2 dx = \frac{1}{2+1}x^{2+1} + C = \frac{1}{3}x^3 + C.$$

Consider that $x = 1$, $y = -2$, hence

$$-2 = \frac{1}{3}(1)^3 + C \iff C = -2 - \frac{1}{3} = -\frac{7}{3}$$

i.e., $y = \frac{1}{3}x^3 - \frac{7}{3}$.

Initial Value Problems

Example

The acceleration of a falling particle near the surface of the earth is approximately $g = 9.8\text{m}/\text{s}^2$. If $v(t)$ is the velocity of the particle, and the initial value at $t = 0$ is $v(0) = v_0$, find $v(t)$.

The initial value problem is: $\dfrac{dv}{dt} = -g$, and $v(0) = v_0$. We have

$$v(t) = -\int g dt = -gt + C.$$

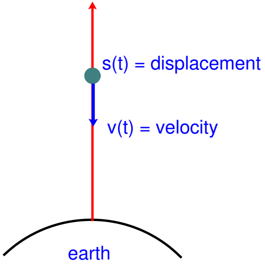
Putting in $t = 0$, we have $v_0 = -9.8(0) + C = C$ i.e., $v(t) = -gt + v_0$.

Initial Value Problems

If $s(t)$ is the displacement function in above example, and the initial position is given as $s(0) = s_0$, then $\frac{ds}{dt} = v(t)$, and

$$s(t) = \int v(t)dt = \int (-gt + v_0)dt = -\frac{1}{2}gt^2 + v_0 t + C$$

Putting in $s(0) = s_0$, we have $C = s_0$ and $s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$.



Example

The acceleration of a particle moving along a line is given by $a = 2t + 1$. If the initial position of the particle is $s(0) = 4$ and initial velocity $v(0) = -2$, find the position function of the particle. (all quantities are in SI units)

Note that $\frac{dv}{dt} = a = 2t + 1$, hence

$$v(t) = \int (2t+1)dt = t^2 + t + C$$

Putting in $t = 0$, we have

$$-2 = v(0) = (0)^2 - 0 + C \Longleftrightarrow C = -2$$

i.e., $v(t) = t^2 + t - 2$. Since $s'(t) = v(t)$, we have

$$s(t) = \int (t^2 + t - 2)dt = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + C_1.$$

Putting in $t = 0$, we have $4 = 0 + C_1 = C_1$ and $s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + 4$

Outline

Area Under a Graph By Squeezing

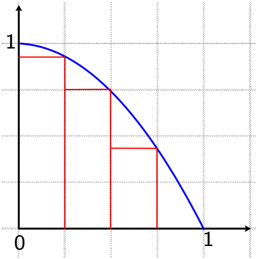
The idea of definite integral is basically from a combination of the approximation of area by rectangles and limit taking.

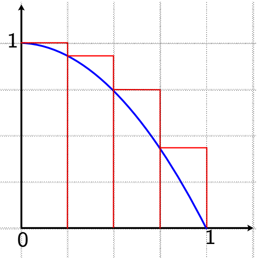
Let's start with a simple example to illustrate how the area under the graph of a function can be squeezed out by using rectangular approximations of the region.

Area Under $y = 1 - x^2$ over $[0, 1]$

Problem: Finding the area A under the graph of $y = 1 - x^2$ over the interval $[0, 1]$.

Although it is not immediately clear what the area A is, it is extremely easy to estimate the area A roughly by using just a few rectangles.



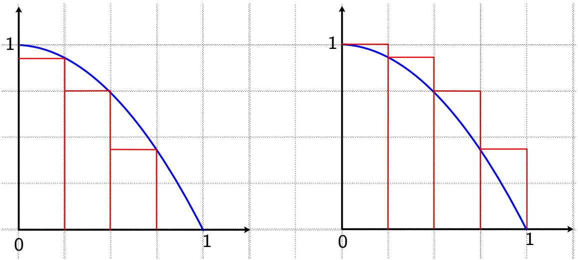


Area Under $y = 1 - x^2$ over $[0, 1]$

1. Divide the interval $[0, 1]$ into 4 subintervals of the same length.
2. Use rectangles based on these subintervals, with heights equal to functions values at the left or right endpoints of the subintervals, to estimate the area.
3. The area A is then squeezed between sums of rectangular areas:

$$A > (1 - (0.25)^2) \cdot 0.25 + (1 - (0.5)^2) \cdot 0.25 + (1 - (0.75)^2) \cdot 0.25 = 0.53125$$

$$A < (1 - 0^2) \cdot 0.25 + (1 - (0.25)^2) \cdot 0.25 + (1 - (0.5)^2) \cdot 0.25 + (1 - (0.75)^2) \cdot 0.25 = 0.78125$$



Area Under $y = 1 - x^2$ over $[0, 1]$

By subdividing $[0, 1]$ into more and more subintervals, we expect better and better estimates of the area A, and eventually squeezing out the area by taking limit.

More precisely, we start by subdividing the interval $[0, 1]$ into n subintervals of the same length $\frac{1}{n}$ by the subdivision points

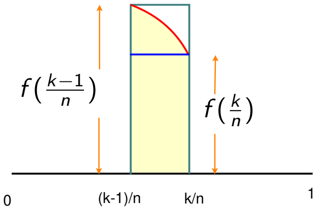
$$0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n}{n} = 1$$

Over the interval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$, $k = 1, \ldots, n$, we have a rectangular area sandwich:

$$\left[1 - \left(\frac{k}{n}\right)^2\right]\frac{1}{n} < A_k < \left[1 - \left(\frac{k-1}{n}\right)^2\right]\frac{1}{n}$$

$$\frac{1}{n} - \frac{k^2}{n^3} < A_k < \frac{1}{n} - \frac{(k-1)^2}{n^3}$$

$A_k =$ shaded area under the graph



Area Under $y = 1 - x^2$ over $[0, 1]$

We can bound the area of k-th parts as follows:

$$\frac{1}{n} - \frac{k^2}{n^3} < A_k < \frac{1}{n} - \frac{(k-1)^2}{n^3}.$$

Adding all these rectangular area sandwiches together, we have

$$\underbrace{n \cdot \frac{1}{n} - \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right)}_{\text{under estimate}(n)} < A < \underbrace{n \cdot \frac{1}{n} - \frac{1}{n^3}\left(0^2 + 1^2 + \cdots + (n-1)^2\right)}_{\text{upper estimate}(n)}.$$

Note that difference of two estimates tends to zero

$$\text{upper estimate}(n) - \text{ lower estimate}(n) = \frac{1}{n} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Area Under $y = 1 - x^2$ over $[0, 1]$

Sandwich theorem means

$$A = \lim_{n\to\infty} \text{upper estimate}(n) = \lim_{n\to\infty} \text{lower estimate}(n).$$

Squeeze Theorem (or Sandwich Theorem)

Let I be an interval having the point a. Let g, f, and h be functions defined on I, except possibly at a itself. Suppose that for every x in I NOT equal to a, we have If $g(x) \le f(x) \le h(x)$ for all x near a, except perhaps when $x = a$, then

$$\lim_{x\to a} g(x) \le \lim_{x\to a} f(x) \le \lim_{x\to a} h(x)$$

whenever these limits exist. (we allow a be ∞ or $-\infty$)

Hence, to find the are under curve, we only needs to find limits

$$\lim_{n\to\infty}\left[n \cdot \frac{1}{n} - \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right)\right] = \lim_{n\to\infty}\left[1 - \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}\right]$$

or

$$\lim_{n\to\infty}\left[n \cdot \frac{1}{n} - \frac{1}{n^3}\left(0^2 + 1^2 + \cdots + (n-1)^2\right)\right] = \lim_{n\to\infty}\left[1 - \frac{1}{n^3}\frac{(n-1)n(2n-1)}{6}\right].$$

Tutorial/Exercise

Show that

$$1^2 + 2^2 + 3^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6}.$$

Area Under $y = 1 - x^2$ over $[0, 1]$

We have

$$\begin{aligned}
&\lim_{n\to\infty} \text{lower estimate}(n) \\
=&\lim_{n\to\infty} \left[n \cdot \frac{1}{n} - \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right)\right] \\
=&\lim_{n\to\infty} \left[1 - \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}\right] \\
=&\lim_{n\to\infty} \left[1 - \frac{(1+\frac{1}{n})(2+\frac{1}{n})}{6}\right] \\
=&\frac{2}{3}
\end{aligned}$$

Area Under $y = 1 - x^2$ over $[0, 1]$

Similarly,

$$\begin{aligned}
&\lim_{n\to\infty} \text{upper estimate}(n) \\
=&\lim_{n\to\infty} \left[n \cdot \frac{1}{n} - \frac{1}{n^3} \left(0^2 + 1^2 + \cdots + (n-1)^2 \right) \right] \\
=&\lim_{n\to\infty} \left[1 - \frac{1}{n^3} \frac{(n-1)n(2n-1)}{6} \right] \\
=&\lim_{n\to\infty} \left[1 - \frac{(1-\frac{1}{n})(2-\frac{1}{n})}{6} \right] \\
=&\frac{2}{3}
\end{aligned}$$

Hence, the area under the graph of $y = 1 - x^2$ over $[0, 1]$ is $\dfrac{2}{3}$.

Area and Displacement

Note that the "area" under the graph of a function $y = 1 - x^2$ over the interval $[0, 1]$ may be used to represent other quantities.

For example, if we consider the velocity function of a particle moving along a line, say, $v(t) = 1 - t^2$, then all those rectangle areas computed in above example is

$$\underbrace{\left[1 - \left(\frac{k}{n}\right)^2\right]}_{velocity\ \ at\ \ t=k/n} \cdot \underbrace{\frac{1}{n}}_{time} < A_k < \left[1 - \left(\frac{k-1}{n}\right)^2\right] \cdot \frac{1}{n}$$

could be used to estimate the displacement of the particle during the time interval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$. Hence the result of the limit calculation, $A = \frac{2}{3}$, is now the displacement of the particle during the time interval $0 \le t \le 1$.

Outline

Riemann Sums

The process in computing area in above example can obviously be applied to any continuous function f on the interval $[a, b]$.

The so called Riemann sum of a continuous function $f(x)$ on an interval $[a, b]$ with respect to a subdivision of the interval into n subintervals by the points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

is a straightforward generalization of rectangular approximation of area.

Riemann Sums

More precisely, denote the length of the i-th subinterval $[x_{i-1}, x_i]$ by Δx_i, and choose for each a point c_i in the subinterval $[x_{i-1}, x_i]$ for each $i = 1, 2, \ldots, n$. The corresponding Riemann sum is defined by:

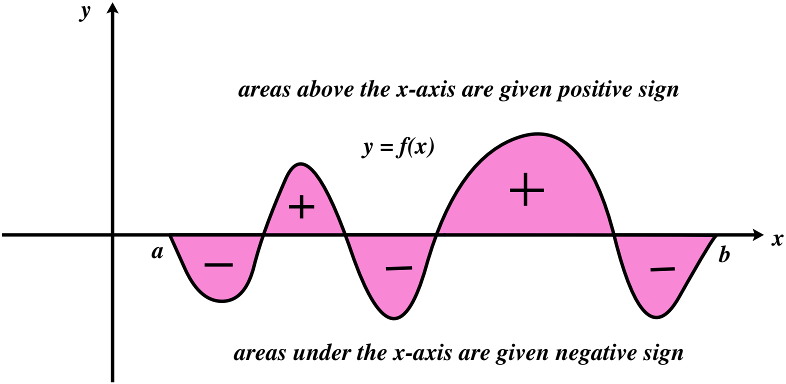
$$S_n = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

1. If $c_i = x_{i-1}$ for all i, then S_n is called a left (left point) Riemann sum.
2. If $c_i = x_i$ for all i, then S_n is called a right Riemann (right point) sum.
3. If $c_i = (x_{i-1} + x_i)/2$ for all i, then S_n is called a middle (middle point) Riemann sum.

For specific function, Riemann sums converge as the partition "gets finer and finer" (n gets larger and larger).

Riemann Sums and Signed Area

If you look at the graph of the function, a Riemann sum is just a rectangular approximation of the **signed area** (+ve/-ve area) between the graph and the x-axis, based on the chosen points x_i's and c_i's.



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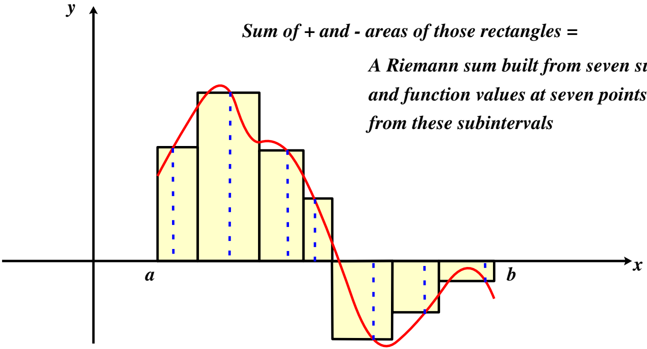


Riemann Sums and Definite Integrals

Taking into account the limiting behaviour of Riemann sums over finer and finer subdivisions, the definite integral of a continuous function $f(x)$ on an interval $[a, b]$ is defined and denoted by

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i \qquad \text{whenever the limit exists.}$$

Geometrically speaking, if area above the x-axis is counted as positive area, and area below the x-axis as negative area, then

$$\int_a^b f(x)dx$$

is the sum of +ve and -ve area of the region between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$.

Riemann Sums and Definite Integrals

Just recall that the "rectangular areas" in the Riemann sum could actually mean certain quantity other than area, e.g., displacement.
The actually meaning of a definite integral

$$\int_a^b f(x)dx$$

in application relies on the meaning on the product $f(c_i)\Delta x_i$, i.e., the unit from "unit of the y-axis" times "unit of the x-axis".

Riemann Sums and Definite Integrals

The summation notation, or the sigma notation, is often used to express the sum of a number of terms indexed by integers:

$$a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

For example: $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2$.

A basic property of the summation notation is: for any constants A, B,

$$\sum_{k=1}^{n} [Aa_k + Bb_k] = A \sum_{k=1}^{n} a_k + B \sum_{k=1}^{n} b_k.$$

Riemann Sums and Definite Integrals

The definite integral of a continuous function $f(x)$ on an interval $[a, b]$ can also be defined in a somewhat simplified but equivalent way, namely, by using subintervals of equal length

$$\Delta x = \frac{b-a}{n};$$

i.e., with subdivision points

$$a = x_0 < x_1 < x_2 < \cdots < x_i < \cdots x_n = b$$

where $x_i = x_0 + i\Delta x$, and c_i in $[x_{i-1}, x_i]$, such that

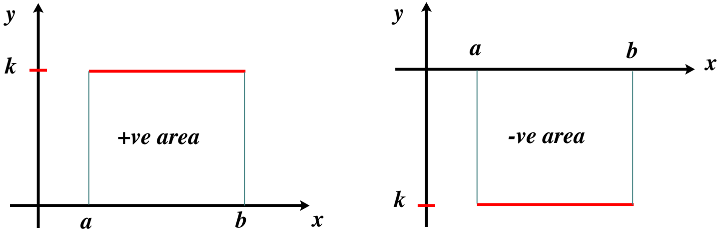
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\frac{b-a}{n} \qquad \text{whenever the limit exists}$$

Example: $\int_a^b k dx = k(b-a)$

An easy example: $f(x) = k$ where k is a constant.

Of course, by area consideration, we expect

$$\int_a^b f(x)dx = \int_a^b k dx = k(b-a)$$



Let's show that according to the Riemann sums.

Example: $\int_a^b kdx = k(b-a)$

Take any subdivision of the interval

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

we have a constant Riemann sum

$$k(x_1 - x_0) + k(x_2 - x_1) + k(x_3 - x_2) + \cdots + k(x_n - x_{n-1}) = k(x_n - x_0) = k(b-a)$$

since $f(c_i) = k$, no matter how you choose c_i in $[x_{i-1}, x_i]$.

The limit of the Riemann sums as $n \to \infty$ is then obviously $k(b-a)$.