Calculus IB: Lecture 18

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Outline

Outline

Antiderivatives

1. Differentiation problem: Given a function $f \longrightarrow$ find $\dfrac{df}{dx}$.
2. Reversing the process: Given a function $f \longrightarrow$ find a function F such that $F' = f$.
3. This can also be considered as a question of solving the "differential equation"

$$\frac{d}{dx}(\text{which function}) = f(x) \ .$$

4. Any function F satisfying $F' = f$ is called an *antiderivative* (or a primitive function) of f.

Antiderivatives

1. Any function F satisfying $F' = f$ is called an *antiderivative* (or a primitive function) of f.
2. Obviously, if F is an antiderivative of f, then so is $F + C$ for any constant C, since $\frac{dC}{dx} = 0$.
3. Note that if F and G are two antiderivatives of f on an open interval, then we have

$$(F - G)' = F' - G' = f - f = 0 \ .$$

By the mean value theorem, $F - G$ must then be a constant function on the interval; i.e., $G(x) - F(x) = C$ for some constant C.

Antiderivatives

Theorem (Mean Value Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some $c \in (a, b)$, or equivalently $f(b) - f(a) = f'(c)(b - a)$.

Let $H(x) = F(x) - G(x)$ defined on $I = (p, q)$ and $H'(x) = 0$ for all x in I. Then for any $p < a < b < q$, we have $f(b) - f(a) = f'(c)(b - a) = 0$ for some $c \in (a, b)$.

Therefore if one antiderivative F has been found for a given function f on an open interval, all antiderivatives of f on the interval can be expressed in the form $F + C$, where C is an arbitrary constant.

Antiderivatives

Example

Let $f(x) = 3x^2$. Solve the antiderivative problem: $\frac{d}{dx}(?) = 3x^2$.

Knowing that

$$\frac{dx^3}{dx} = 3x^2,$$

the antiderivatives of $3x^2$ are given by $x^3 + C$, where C is an arbitrary constant.

Antiderivatives

Example

Let $g(x) = 2\cos x$. Solve the antiderivative problem: $\frac{d}{dx}(?) = 2\cos x$.

Since

$$\frac{d \sin x}{dx} = \cos x,$$

it easy to see that

$$\frac{d(2\sin x)}{dx} = 2\cos x.$$

Hence the antiderivatives of $2\cos x$ are given by $2\sin x + C$, where C is an arbitrary constant.

Example

Let $h(x) = x + e^{2x}$. Solve the antiderivative problem: $\frac{d}{dx}(?) = x + e^{2x}$

Since

$$\frac{dx^2}{dx} = 2x \quad \text{and} \quad \frac{de^{2x}}{dx} = 2e^{2x},$$

we have

$$\frac{d}{dx}\left(\frac{1}{2}x^2 + \frac{1}{2}e^{2x}\right) = x + e^{2x}.$$

Hence, the antiderivatives of $h(x) = x + e^{2x}$ are given by

$$\left(\frac{1}{2}x^2 + \frac{1}{2}e^{2x}\right) + C.$$

Indefinite Integral

The *indefinite integral* notation

$$\int f(x)dx$$

is nothing but a new dress of the antiderivatives! The function $f(x)$ appearing in an indefinite integral is usually called the *integrand*.

Indefinite Integral

For example,

$$\int 3x^2 dx \overset{means}{=} \text{all antiderivatives of } 3x^2$$

$$\overset{thus}{=} x^3 + C \qquad \left(\text{since } \frac{dx^3}{dx} = 3x^2\right)$$

Equivalently, this is the same as saying that the general solution of the differential equation

$$\frac{dy}{dx} = 3x^2$$

is

$$y = x^3 + C.$$

Indefinite Integral

Example

Show that $\int (2x+1)e^{x^2+x}dx = e^{x^2+x} + C$.

This is just another way to say

$$\frac{d}{dx}(e^{x^2+x}) = e^{x^2+x} \cdot \frac{d(x^2+x)}{dx} = (2x+1)e^{x^2+x}.$$

$(2x+1)e^{x^2+x}$ is the derivative of e^{x^2+x} $\overset{vs}{\Longleftrightarrow}$ e^{x^2+x} is an antiderivative of $(2x+1)e^{x^2+x}$

Indefinite Integral

In fact, we have

$$\int f(x)dx = F(x) + C \Longleftrightarrow \frac{dF}{dx} = f(x).$$

In particular,

$$\frac{d}{dx}\int f(x)dx = f(x),$$

and

$$\int f'(x)dx = f(x) + C.$$

Indefinite Integral

$$\frac{d}{dx}\frac{1}{p+1}x^{p+1} = x^p \quad \stackrel{p\neq -1}{\Longleftrightarrow} \quad \int x^p dx = \frac{1}{p+1}x^{p+1} + C$$

$$\frac{d}{dx}e^x = e^x \quad \Longleftrightarrow \quad \int e^x dx = e^x + C$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \quad \Longleftrightarrow \quad \int \frac{1}{x}dx = \ln|x| + C$$

$$\frac{d}{dx}\sin x = \cos x \quad \Longleftrightarrow \quad \int \cos x dx = \sin x + C$$

$$\frac{d}{dx}[-\cos x] = \sin x \quad \Longleftrightarrow \quad \int \sin x dx = -\cos x + C$$

$$\frac{d}{dx}\tan x = \sec^2 x \quad \Longleftrightarrow \quad \int \sec^2 x dx = \tan x + C$$

$$\vdots$$

Note that

$$\int \frac{1}{x} dx \neq \ln x + C,$$

since the domain of $\ln x$ is $(0, \infty)$, rather than all real numbers.

Exercise

Check the formula

$$\frac{d}{dx} \ln |x| = \frac{1}{x}$$

and

$$\int \frac{1}{x} dx = \ln |x| + C.$$

Indefinite Integral

The term-by-term differentiation rule

$$(aF(x) + bG(x))' = aF'(x) + bG'(x),$$

where a, b are constants, can be rewritten as an integration rule:

$$\int (af(x) + bg(x))dx = a \int f(x)dx + b \int g(x)dx$$

Just note that derivatives of both sides are equal to

$$af(x) + bg(x);$$

i.e., both sides are antiderivatives of

$$af(x) + bg(x).$$

Indefinite Integral

Example

1. $\int (3x^4 - 2x^3)\, dx = 3\int x^4 dx - 2\int x^3 dx = \frac{3}{5}x^5 - \frac{1}{2}x^4 + C$

2. $\int \left(x^5 - e^x + \frac{1}{x}\right) dx = \int x^5 dx - \int e^x dx + \int \frac{1}{x} dx$

   $= \frac{1}{6}x^6 - e^x + \ln|x| + C$

3. $\int (3\sin x - 2\sec^2 x + 3e^x)\, dx = -3\cos x - 2\tan x + 3e^x + C$

Outline

The Substitution Rule

So far, most of our integrals could be found directly, up to some algebraic manipulations.

When an integral looks complicated, without an obvious antiderivative, quite often we need to make it simpler by using a suitable "substitution".

The Substitution Rule

If you know

$$\frac{d}{dx}\sin x^2 = 2x\cos x^2,$$

then it is of course straightforward to write

$$\int 2x\cos x^2 dx = \sin x^2 + C.$$

What if you don't know the derivative?

Let $u = x^2$, so that

$$\frac{du}{dx} = 2x.$$

Now, by formally writing $du = 2xdx$ and putting everything into the original integral, we have

$$\int 2x\cos x^2 dx = \int \cos u\, du \overset{easy}{=} \sin u + C = \sin x^2 + C$$

The Substitution Rule

By letting $u = g(x)$ (some way to group some perhaps complicated x expression as u), and $du = g'(x)dx$, the following may happen:

$$\text{Apply } u = g(x) \text{ and } du = g'(x)dx \text{ on } \int f(x)dx$$

$$\Longrightarrow \text{an easier } u\text{-integral } \int F(u)du$$

$$\Longrightarrow \text{put } u = g(x) \text{ back after finishing the } u\text{-integral}$$

The Substitution Rule

The reason behind the Substitution Rule is Chain Rule!

The chain rule means

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x).$$

By letting $u = g(x)$, $du = g'(x)dx$, we have

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u) + C.$$

Then $F'(u) = f(u)$ and

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)dx$$

and hence the antiderivatives of $f(g(x))g'(x)$ are given by

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

The Substitution Rule

Theorem

If $u = g(x)$ is a differentiable function whose range is an interval I, and $f(x)$ is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

The Substitution Rule

Example

Find the indefinite integral of $\int \sqrt{3x+2}dx$.

Let $u = 3x + 2$ such that

$$\frac{du}{dx} = 3 \text{ and } \frac{1}{3}du = dx.$$

Hence

$$\begin{aligned}\int \sqrt{3x+2}dx =& \int \frac{1}{3}u^{1/2}du \\ =& \frac{1}{3} \cdot \frac{2}{3}u^{3/2} + C \\ =& \frac{2}{9}(3x+2)^{3/2} + C\end{aligned}$$

The Substitution Rule

Example

Find the indefinite integral of $\int \sin(5x+2)dx$.

Let $u = 5x + 2$ such that

$$\frac{du}{dx} = 5 \quad \text{and} \quad \frac{1}{5}du = dx.$$

Hence

$$\begin{aligned}\int \sin(5x+2)dx &= \int \frac{1}{5}\sin u du \\ &= -\frac{1}{5}\cos u + C \\ &= -\frac{1}{5}\cos(5x+2) + C\end{aligned}$$

The Substitution Rule

Example

Find the indefinite integral of $\int \frac{1}{2x+1} dx$.

Let $u = 2x + 1$ such that

$$\frac{du}{dx} = 2 \quad \text{and} \quad \frac{1}{2} du = dx.$$

Hence

$$\begin{aligned}\int \frac{1}{2x+1} dx =& \frac{1}{2} \int \frac{1}{u} du \\ =& \frac{1}{2} \ln |u| + C \\ =& \frac{1}{2} \ln |2x+1| + C\end{aligned}$$

The Substitution Rule

Example

Find the indefinite integral of $\int x^2 e^{x^3+1} dx$.

Let $u = x^3 + 1$ such that

$$\frac{du}{dx} = 3x^2 \quad \text{and} \quad \frac{1}{3} du = x^2 dx.$$

Hence

$$\begin{aligned} \int x^2 e^{x^3+1} dx &= \frac{1}{3} \int e^u du \\ &= \frac{1}{3} e^u + C \\ &= \frac{1}{3} e^{x^3+1} + C \end{aligned}$$

Example

Find the indefinite integral of $\int \frac{3t}{4t^2 - 1} dt$.

Let $u = 4t^2 - 1$ such that

$$\frac{du}{dt} = 8t \quad \text{and} \quad \frac{1}{8} du = t dt.$$

Hence

$$\begin{aligned} \int \frac{3t}{4t^2 - 1} dt &= \frac{3}{8} \int \frac{1}{u} du \\ &= \frac{3}{8} \ln |u| + C \\ &= \frac{3}{8} \ln |4t^2 - 1| + C \end{aligned}$$

The Substitution Rule

Exercise

Find the following indefinite integral

1. $\displaystyle\int \frac{x}{\sqrt{x^2+1}}dx$
2. $\displaystyle\int \sec x dx \quad (\text{hint: let } u = \sec x + \tan x)$
3. $\displaystyle\int \sin^3\theta d\theta$
4. $\displaystyle\int \cos^3\theta d\theta$
5. $\displaystyle\int x e^{6x^2} dx$

Outline

Integration by Parts

Let's study the integral

$$\int x\mathrm{e}^{6x}\,dx.$$

If the integrand was e^{6x^2}, we could do the integral with a substitution $u = x^2$. Unfortunately, such idea does not work here.

To do this integral we will need to use integration by parts.

Integration by Parts

We'll start with the product rule of derivative

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

Now integrate both sides of this formula

$$\begin{aligned} f(x)g(x) &= \int [f(x)g(x)]'\, dx \\ &= \int \left[f'(x)g(x) + f(x)g'(x)\right] dx \\ &= \int f'(x)g(x)dx + \int f(x)g'(x)dx \end{aligned}$$

Substituting $u = f(x)$, $v = g(x)$, $du = f'(x)dx$ and $dv = g'(x)dx$, then

$$\int udv = uv - \int vdu$$

The following formula is called integration by parts

$$\int u dv = uv - \int v du$$

To use this formula, we will need to identify u and dv, then compute

$$\int v du$$

Example

Find the indefinite integral of $\int xe^{6x}dx$.

Let $u = x$ and $dv = e^{6x}dx$, then $du = dx$ and

$$v = \int e^{6x}dx = \frac{1}{6}e^{6x}$$

Using integration by parts, we have

$$\begin{aligned}\int xe^{6x}dx = \int udv = uv - \int vdu \\ = \frac{x}{6}e^{6x} - \int \frac{1}{6}e^{6x}dx = \frac{x}{6}e^{6x} - \frac{1}{36}e^{6x} + C\end{aligned}$$

Integration by Parts

Example

Find the indefinite integral of $\int (3t+5)\cos\left(\frac{t}{4}\right) dt$.

Let $u = 3t+5, dv = \cos\left(\frac{t}{4}\right) dt, du = 3dt, v = 4\sin\left(\frac{t}{4}\right)$.

Using integration by parts, we have

$$\int (3t+5)\cos\left(\frac{t}{4}\right) dt = \int udv = uv - \int vdu$$
$$=4(3t+5)\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right)(3dt)$$
$$=4(3t+5)\sin\left(\frac{t}{4}\right) - 12\int \sin\left(\frac{t}{4}\right) dt$$
$$=4(3t+5)\sin\left(\frac{t}{4}\right) + 48\cos\left(\frac{t}{4}\right) + C$$

Integration by Parts

Exercise

Find the following indefinite integral

1. $\int x\sqrt{x+1}dx$

2. $\int \ln x dx$