Calculus IB: Lecture 16

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Outline

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One may use derivatives to help compute limits of the $\frac{0}{0}$-type or $\frac{\infty}{\infty}$-type, which is essentially what L'Hôpital's rule does.

The rule is named after the 17th-century French mathematician Guillaume de L'Hôpital, however, the theorem was first introduced in 1694 by the Swiss mathematician Johann Bernoulli.

Johann Bernoulli (1667–1748) Guillaume de L'Hôpital (1661–1704)

L'Hôpital's Rule

Roughly speaking, if $f(a) = g(a) = 0$, we can use the idea of linear approximation to find the limit of $f(x)/g(x)$:

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)} &\approx \lim_{x\to a}\frac{f(a)+f'(a)(x-a)}{g(a)+g'(a)(x-a)}\\ &= \lim_{x\to a}\frac{f'(a)(x-a)}{g'(a)(x-a)}\\ &= \frac{f'(a)}{g'(a)}.\end{aligned}$$

In some appropriate conditions, we have

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Baby L'Hôpital's Rule

Theorem (Baby L'Hôpital's Rule, $\frac{0}{0}$-type)

Let $f(x)$ and $g(x)$ be continuous functions on an interval containing $x = a$, with $f(a) = g(a) = 0$. Suppose that f and g are differentiable, and f' and g' are continuous. Finally, suppose that $g'(a) \neq 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}.$$

We also have

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)} \quad \textit{and} \quad \lim_{x\to a^-} \frac{f(x)}{g(x)} = \lim_{x\to a^-} \frac{f'(x)}{g'(x)}.$$

Baby L'Hôpital's Rule

Proof.

Since $f(a) = g(a) = 0$, we have

$$\lim_{x\to a}\frac{f(a)}{g(a)} = \lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a}\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}$$

$$=\frac{\lim\limits_{x\to a}\dfrac{f(x)-f(a)}{x-a}}{\lim\limits_{x\to a}\dfrac{g(x)-g(a)}{x-a}} = \frac{f'(a)}{g'(a)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

where the last step use the continuity of f' and g'. □

Theorem (Macho L'Hôpital's Rule, one-side)

Suppose that f and g are continuous on a closed interval $[a, b]$, and are differentiable on the open interval (a, b). Suppose that $g'(x)$ is never zero on (a, b) and $\lim_{x\to a^+} \frac{f'(x)}{g'(x)}$ exists, and that $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$. Then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

This theorem doesn't require anything about $g'(a)$, just about how g' behaves to the right of a.

The conclusion relates limit of $f(x)/g(x)$ to another one-side limit of $f(x)'/g(x)'$, and not to the value of $f'(a)/g'(a)$.

Macho L'Hôpital's Rule

Exercise/Tutorial

Prove Macho L'Hôpital's rule.

Hint: Although we do not suppose what is $f(a)$ and $g(a)$, we can define

$$F(x) = \begin{cases} 0 & x = a \\ f(x) & x > a \end{cases} \quad \textit{and} \quad G(x) = \begin{cases} 0 & x = a \\ g(x) & x > a \end{cases}.$$

Then try to prove the following theorem and apply it.

Theorem (cauchy's mean value theorem)

If $F(x)$ and $G(x)$ are continuous on $[a, b]$ and differentiable on (a, b), then there is a point c in (a, b) such that

$$(F(b) - F(a))G'(c) = (G(b) - G(a))F'(c).$$

(when $G(x) = x$, this is the same as the usual mean value theorem)

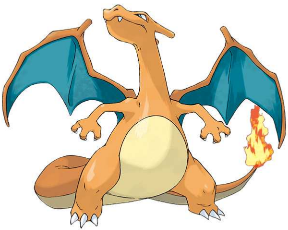
General form of L'Hôpital's Rule

The previous versions apply to forms of type $\dfrac{\infty}{\infty}$ as well as $\dfrac{0}{0}$, and apply to limits as $x \to \infty$ or $x \to -\infty$ as well as to limits $x \to a^+$ or $x \to a^-$. In all of these cases, the rule is:

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}.$$

baby form

macho form



general form

General form of L'Hôpital's Rule

Theorem (General form of L'Hôpital's Rule)

Let c and L be extended real numbers (i.e., real numbers, positive infinity, or negative infinity). Let I be an open interval containing c (for two-sided limit) or an open interval with endpoint c (for one-sided limit, or a limit at infinity if c is infinite). The real valued functions f and g are assumed to be differentiable on I except possibly at c, and additionally $g'(x) \neq 0$ on I except possibly at c. It is also assumed that

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L.$$

If either $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\lim_{x \to c} |f(x)| = \lim_{x \to c} |g(x)| = \infty$, then $\lim_{x \to c} \dfrac{f(x)}{g(x)} = L$. The limits also can be one-sided limits $x \to c^+$ or $x \to c^-$, when c is a finite endpoint of I.

Examples of L'Hôpital's Rule

Example ($\frac{0}{0}$-type)

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\frac{d\sin x}{dx}}{\frac{dx}{dx}}=\lim_{x\to 0}\frac{\cos x}{1}=\cos 0=1.$$

However, this computation is somewhat stupid, because we have used

$$\frac{d\sin x}{dx}=\cos x.$$

We can directly obtain the result by the definition of derivative

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\sin x-\sin 0}{x-0}=\frac{d\sin x}{dx}\bigg|_{x=0}=\cos 0=1.$$

Examples of L'Hôpital's Rule

Example ($\frac{0}{0}$-type)

1. $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3} = \lim\limits_{x\to 0} \dfrac{1-\cos x}{3x^2} = \lim\limits_{x\to 0} \dfrac{\sin x}{6x} = \dfrac{1}{6}.$
2. $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\sec x}{1+\tan x} = \lim\limits_{x\to \frac{\pi}{2}} \dfrac{\sec x \tan x}{\sec^2 x} = \lim\limits_{x\to \frac{\pi}{2}} \sin x = 1.$
3. $\lim\limits_{x\to 0^+} \dfrac{\sin x}{x^2} = \lim\limits_{x\to 0^+} \dfrac{\cos x}{2x} = +\infty.$
4. $\lim\limits_{x\to 0^-} \dfrac{\sin x}{x^2} = \lim\limits_{x\to 0^-} \dfrac{\cos x}{2x} = -\infty.$

Examples of L'Hôpital's Rule

Example ($\frac{0}{0}$-type)

1. $$\lim_{x\to 2}\frac{x^2-4}{x^3-8}=\lim_{x\to 2}\frac{\frac{d}{dx}(x^2-4)}{\frac{d}{dx}(x^3-8)}=\lim_{x\to 2}\frac{2x}{3x^2}=\frac{2\cdot 2}{3\cdot 2^2}=\frac{1}{3}$$

2. $$\lim_{x\to 0}\frac{x}{\sqrt{3x+4}-2}=\lim_{x\to 0}\frac{\dfrac{dx}{dx}}{\dfrac{d[(3x+4)^{1/2}-2]}{dx}}$$

$$=\lim_{x\to 0}\frac{1}{\frac{3}{2}(3x+4)^{-1/2}}=\lim_{x\to 0}\frac{2}{3}(3x+4)^{1/2}=\frac{4}{3}$$

Example ($\frac{\infty}{\infty}$-type)

1. $\lim_{x\to\infty}(x^2e^{-x}) = \lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0.$

   In general $\lim_{x\to+\infty}\frac{p(x)}{e^x} = 0$ for any polynomial $p(x)$.

2. $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(-x) = 0$

3. $\lim_{x\to+\infty}\frac{\ln x}{x} = \lim_{x\to+\infty}\frac{\frac{d}{dx}\ln x}{\frac{dx}{dx}} = \lim_{x\to+\infty}\frac{\frac{1}{x}}{1} = 0$

Examples of L'Hôpital's Rule

Example ($\frac{\infty}{\infty}$-type)

1. $\lim\limits_{x\to+\infty} x^{\frac{1}{x}} = \lim\limits_{x\to+\infty} e^{\ln x^{\frac{1}{x}}} = \lim\limits_{x\to+\infty} e^{\frac{1}{x}\ln x} = e^{\lim\limits_{x\to+\infty}\frac{\ln x}{x}} = e^0 = 1$, since
$$\lim_{x\to+\infty}\frac{\ln x}{x} = 0.$$

2. $\lim\limits_{x\to+\infty}\left(1-\dfrac{4}{x}\right)^x = \lim\limits_{x\to+\infty} e^{\ln\left(1-\frac{4}{x}\right)^x} = \lim\limits_{x\to+\infty} e^{x\ln(1-\frac{4}{x})} = e^{-4}$, since
$$\lim_{x\to+\infty}\frac{\ln(1-\frac{4}{x})}{\frac{1}{x}} = \lim_{x\to+\infty}\frac{\frac{\frac{4}{x^2}}{1-\frac{4}{x}}}{-\frac{1}{x^2}} = \lim_{x\to+\infty}\frac{-4x}{x-4} = -4$$

Outline

Common Mistakes when Using L'Hôpital's Rule

L'Hôpital's rule compute $\frac{0}{0}$ or $\frac{\infty}{\infty}$ type limit, it doesn't solve every limit.

Example (Incorrect Application of L'Hôpital's rule)

Try to use L'Hôpital's rule to evaluate $\lim_{x \to 3} \frac{2x+7}{4x+1}$.

Apply L'Hôpital's rule to the limit and then evaluate.

$$\lim_{x \to 3} \frac{2x+7}{4x+1} = \lim_{x \to 3} \frac{\frac{d}{dx}(2x+7)}{\frac{d}{dx}(4x+1)} = \lim_{x \to 3} \frac{2}{4} = \frac{1}{2}$$

However, this is incorrect! The function near $x = 3$ is continuous and the actual limit value should be 1.

What is wrong with it?

Both $\lim_{x \to 3}(2x+7)$ and $\lim_{x \to 3}(4x+1)$ are not 0 or ∞.

WARNING: L'Hôpital's rule is NOT a universal tool

Example (Failure of L'Hôpital's rule)

Try to use L'Hôpital's rule to evaluate $\lim_{x\to\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

We first check the condition of L'Hôpital's rule

$$\lim_{x\to\infty}(e^x - e^{-x}) = \infty - 0 = \infty, \quad \lim_{x\to\infty}(e^x + e^{-x}) = \infty + 0 = \infty.$$

Hence it has the form of $\frac{\infty}{\infty}$. We apply L'Hôpital's rule:

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{\frac{d}{dx}(e^x - e^{-x})}{\frac{d}{dx}(e^x + e^{-x})} = \lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{\infty}{\infty}$$

We can easily see that repeated applications of L'Hôpital's rule will just result in the function flipping over and over.

WARNING: L'Hôpital's rule is NOT a universal tool

Example (Failure of L'Hôpital's rule)

It is easy to evaluate $\lim_{x\to\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$ by simplifying step at first.

We just need to multiply $\dfrac{e^{-x}}{e^{-x}}$ as follows:

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{(e^x - e^{-x})}{(e^x + e^{-x})} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1-0}{1+0} = 1.$$

WARNING: L'Hôpital's rule is NOT a universal tool

Example (Failure of L'Hôpital's rule)

Try to use L'Hôpital's rule to evaluate $\lim\limits_{x\to\infty} \dfrac{\sqrt{4x^2+3}}{x+3}$.

It is easy to see the limit has $\dfrac{\infty}{\infty}$ form. Using L'Hôpital's rule, we have

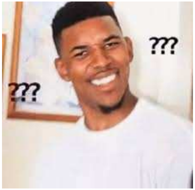
$$
\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{4x^2+3}}{x+3} &= \lim_{x\to\infty} \frac{(4x^2+3)^{1/2}}{x+3} && \text{Rewriting the original limit} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}(4x^2+3)^{1/2}}{\frac{d}{dx}(x+3)} && \text{L'Hôpital's rule} \\
&= \lim_{x\to\infty} \frac{\frac{1}{2}(4x^2+3)^{-1/2}\cdot 8x}{1} \\
&= \lim_{x\to\infty} \frac{4x}{(4x^2+3)^{1/2}} = \frac{\infty}{\infty}.
\end{aligned}
$$

WARNING: L'Hôpital's rule is NOT a universal tool

Example (Failure of L'Hôpital's rule)

Apply L'Hôpital's a second time, and re-evaluate the limit.

$$\begin{aligned}
\lim_{x\to\infty} \frac{4x}{(4x^2+3)^{1/2}} &= \lim_{x\to\infty} \frac{\frac{d}{dx}(4x)}{\frac{d}{dx}(4x^2+3)^{1/2}} && \text{L'Hôpital's rule} \\
&= \lim_{x\to\infty} \frac{4}{\frac{1}{2}(4x^2+3)^{-1/2}\cdot 8x} \\
&= \lim_{x\to\infty} \frac{(4x^2+3)^{1/2}}{x} = \frac{\infty}{\infty} && \text{L'Hôpital's rule again?}
\end{aligned}$$



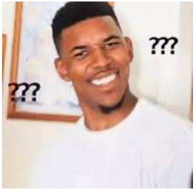
WARNING: L'Hôpital's rule is NOT a universal tool

Example (Failure of L'Hôpital's rule)

$$\cdots = \lim_{x\to\infty} \frac{(4x^2+3)^{1/2}}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(4x^2+3)^{1/2}}{\frac{d}{dx}(x)}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2}(4x^2+3)^{-1/2}\cdot 8x}{1} = \lim_{x\to\infty} \frac{4x}{(4x^2+3)^{1/2}}$$

This is exactly the function we got at the start of last page! No matter how many times we try using L'Hôpital's's rule, it will never yield a limit value. The process will just keep cycling around and around.



WARNING: L'Hôpital's rule is NOT a universal tool

L'Hôpital's rule guarantees that

$$\underbrace{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}_{\text{if the limits exist, the limits are equal.}}$$

However, it does not guarantee the second limit can be evaluated.

Most of the time the second (or third, or fourth, or ...) limits we get from the rule have a simpler form and the new limit is more easily evaluated. But as this example shows, that isn't always the case.

Instead of repeating L'Hôpital's rule, it is important (even critical) to simplify the function at each stage.

WARNING: L'Hôpital's rule is NOT a universal tool

Example (Failure of L'Hôpital's rule)

Factor the x^2 out of the square-root:

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+3}}{x+3} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(4+\frac{3}{x^2}\right)}}{x+3} = \lim_{x\to\infty} \frac{\sqrt{x^2}\cdot\sqrt{4+\frac{3}{x^2}}}{x+3}$$

$$= \lim_{x\to\infty} \frac{|x|\cdot\sqrt{4+\frac{3}{x^2}}}{x+3} = \lim_{x\to\infty} \frac{x\cdot\sqrt{4+\frac{3}{x^2}}}{x+3}$$

Since $x\to\infty$, we replace $|x|$ with just x. Factor the x out of the denominator:

$$\lim_{x\to\infty} \frac{x\cdot\sqrt{4+\frac{3}{x^2}}}{x+3} = \lim_{x\to\infty} \frac{x\cdot\sqrt{4+\frac{3}{x^2}}}{x\left(1+\frac{3}{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{4+\frac{3}{x^2}}}{1+\frac{3}{x}} = \frac{\lim\limits_{x\to\infty}\sqrt{4+\frac{3}{x^2}}}{\lim\limits_{x\to\infty}\left(1+\frac{3}{x}\right)} = 4.$$

L'Hôpital's Rule

1. L'Hôpital's rule help us compute limits of the $\frac{0}{0}$ or $\frac{\infty}{\infty}$-type
2. L'Hôpital's rule is not a universal tool.
3. We must check the form of limit before applying L'Hôpital's rule.
4. Sometimes, simplifying the expression is more useful.

Outline

Convex Optimization

We study how to find the minimum of a convex function f.

This section is beyond the requirement of MATH 1013. It will NOT been contained in our homework or exam, however, it is very helpful to understand the concepts of convex function, linear approximation, optimization problem and Newton methods (next week).

We introduce the following assumptions:

1. the domain of f is $(-\infty, \infty)$
2. f is twice differentiable
3. for any x, we have $f''(x) \leq L$ for some positive constant L
4. there exists x^* such that $f(x^*)$ is the minimum

Note that there could be no closed form expression of x^* or $f(x^*)$.

Hence, we desire to generate the sequence

$$x_0, x_1, x_2, x_3 \ldots$$

such that $|x_k - x^*|$ or $|f(x_k) - f(x^*)|$ convergences to 0 with increasing k.

Exercise

If f is twice differentiable on (a, b) and continuous on $[a, b]$, then

$$f(b) - f(a) - f'(a)(b-a) = \frac{f''(c)}{2}(b-a)^2$$

for some $c \in (a, b)$.

Let $a = x_k$ and $b = x \neq x_k$, then there exists c such that

$$\begin{aligned} f(x) =& f(x_k) + f'(x_k)(x - x_k) + \frac{f''(c)}{2}(x - x_k)^2 \\ \leq& f(x_k) + f'(x_k)(x - x_k) + \frac{L}{2}(x - x_k)^2 \end{aligned}$$

The inequality

$$f(x) \le f(x_k) + f'(x_k)(x - x_k) + \frac{L}{2}(x - x_k)^2$$

provides an upper bound of $f(x)$.

By fixing x_k, the minimizer of the upper bound

$$g_k(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{L}{2}(x - x_k)^2$$

has closed form solution! We have $g_k'(x) = f'(x_k) + L(x - x_k)$, then

$$g_k'(x) = 0 \Longrightarrow f'(x_k) + L(x - x_k) = 0 \Longrightarrow x = x_k - \frac{1}{L}f'(x_k)$$

We can select any x_0 as initial point and run iteration

$$x_{k+1} = x_k - \frac{1}{L}f'(x_k) \quad \text{with } k = 1, 2 \ldots ....$$

For any initial point x_0, the iteration scheme

$$x_{k+1} = x_k - \frac{1}{L} f'(x_k) \quad \text{with } k = 1, 2 \ldots \ldots.$$

satisfies

$$f(x_k) - f(x^*) \leq \frac{|x_0 - x^*|}{2Lk}.$$

Since $|x_0 - x^*|$ and L are independent to k, we have

$$\lim_{k \to \infty} \frac{|x_0 - x^*|}{2Lk} = 0.$$

Hence, $f(x_k)$ convergences to $f(x^*)$.

If we additionally suppose f is strongly convex with constant c.

Then for any initial point x_0, the iteration scheme

$$x_{k+1} = x_k - \frac{1}{L} f'(x_k) \quad \text{with } k = 1, 2 \ldots \ldots$$

satisfies

$$f(x_k) - f(x^*) \leq \left(1 - \frac{c}{L}\right)^k (f(x_0) - f(x^*)).$$

In fact, we can prove $c \leq L$ which means

$$\lim_{k\to\infty} \left(1 - \frac{c}{L}\right)^k (f(x_0) - f(x^*)) = 0.$$

Hence, $f(x_k)$ convergences to $f(x^*)$.

By comparing the convergence result

$$f(x_k) - f(x^*) \leq \frac{|x_0 - x^*|}{2Lk} \quad \text{for convex } f$$

and

$$f(x_k) - f(x^*) \leq \left(1 - \frac{c}{L}\right)^k (f(x_0) - f(x^*)) \quad \text{for strongly convex } f,$$

we have

$$\lim_{k\to\infty} \frac{\left(1 - \frac{c}{L}\right)^k (f(x_0) - f(x^*))}{\frac{|x_0 - x^*|}{2Lk}} = 0.$$

The iteration

$$x_{k+1} = x_k - \frac{1}{L} f'(x_k) \quad \text{with } k = 1, 2 \dots \dots$$

is called gradient descent, which is the most popular method to find the minimum of a convex function.

A convex function on a closed interval I must have a maximum on the endpoint of I.