Calculus IB: Lecture 13

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Outline

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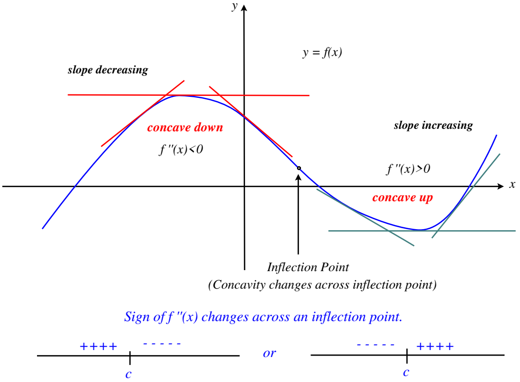
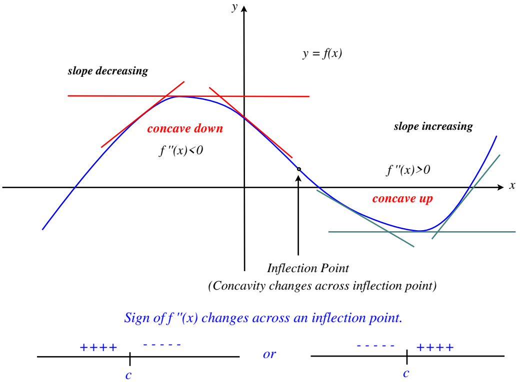
What does the graph of $y = f(x)$ on an interval mean by the sign of f''?

- $f'' > 0 \implies f'$ is increasing (the slope of tangent line is increasing) $\implies f$ is concave up (or strictly convex)
- $f'' < 0 \implies f'$ is decreasing (the slope of tangent line is decreasing). $\implies f$ is concave down (or strictly concave)



If concavity (up/down) on both sides of a point $(c, f(c))$ on the graph of the function $y = f(x)$, where f is continuous, are different, then the point is called a point of inflection.



Convexity/Concavity and 2nd Derivatives

The convexity of f on an interval:

- $f'' \geq 0 \Longrightarrow f$ is convex
- $f'' > 0 \Longrightarrow f$ is strictly convex (concave up)
- $f'' \geq c$ for some $c > 0 \Longrightarrow f$ is strongly convex

The concavity of f on an interval:

- $f'' \leq 0 \Longrightarrow f$ is concave
- $f'' < 0 \Longrightarrow f$ is strictly concave (concave down)
- $f'' \leq c$ for some $c < 0 \Longrightarrow f$ is strongly concave

The linear function is both convex and concave. Since we have $f''(x) = 0$ for $f(x) = ax + b$.

A strictly convex function may NOT be a strongly convex function.

Consider the function $f(x) = e^x$ defined on $(-\infty, \infty)$, we have

$$f'(x) = e^x, \quad f''(x) = e^x \quad \text{and} \quad \lim_{x \to -\infty} f''(x) = 0.$$

Since $f''(x) > 0$, the function $f(x)$ is strictly convex on $(-\infty, \infty)$.

However, for any $c > 0$, there exists M such that $f''(x) < c$ for any $x < M$. Hence, $f(x)$ is not strongly convex.

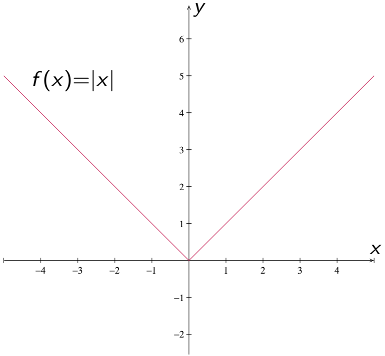
Now we consider the function $f(x) = e^x$ defined on $[a, \infty)$, then

$$f''(x) = e^x \geq e^a$$

By taking $c = e^a$, we have $f''(x) \geq c$ for any x in $[a, \infty)$, which means the function is strongly convex.

Convex/Concave Functions

We can also define of convexity/concavity for non-differentiable functions.



A real-valued function defined on an interval is called convex (concave) if the line segment between any two points on the graph of the function lies above (below) the graph between the two points.

Outline

Graph Sketching

Roughly speaking, 1st/2nd derivatives of a function f, together with some limits (asymptotes), symmetric properties, and intercepts, can help determine the shape of the graph of f pretty well.

We can also use software (e.g. MATLAB) to plot the figure of a function (not allowed in final exam).

Graph Sketching

The following strategies can be used in final exam:

1. Identify the domain of f, and any symmetry property of the graph: e.g., even, odd function?,. . . .
2. Identify the asymptotes, either vertical or horizontal.
3. Compute the first and second derivative : f', f''.
4. Determine the critical points (where is $f'(x) = 0$, or $f'(x)$ does not exist), and interval of increase/decrease (i.e., where does f have positive rate of change $f'(x) > 0$, or negative rate of change $f'(x) < 0$), by looking at the sign line of f'.
5. Determine the concavity of the graph by the sign line of f'', and indicate the inflection points.
6. Plot a suitable number of points, especially the x and/or y intercept, local max/min points, inflection points.

Sketch the graph of the following function

$$y = f(x) = \frac{x^2 - 3}{x^3}.$$

The domain is $x \neq 0$. Note that $x = 0$ is a vertical asymptote since

$$\lim_{x \to 0^+} \frac{x^2 - 3}{x^3} = -\infty, \quad \lim_{x \to 0^-} \frac{x^2 - 3}{x^3} = \infty.$$

and $y = 0$ is a horizontal asymptote since

$$\lim_{x \to \infty} \frac{x^2 - 3}{x^3} = 0 = \lim_{x \to -\infty} \frac{x^2 - 3}{x^3}$$

The function is an odd since $f(-x) = -f(x)$.

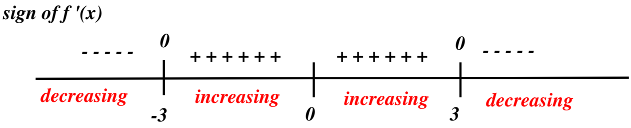
The x-intercept is $x = \pm\sqrt{3}$, since $f(\pm\sqrt{3}) = 0$.

Graph Sketching: Example

Compute the 1-st derivatives and their sign lines:

$$\begin{aligned} f'(x) &= \frac{x^3 \frac{d}{dx}(x^2-3) - (x^2-3)\frac{d}{dx}x^3}{x^6} \\ &= \frac{x^2(9-x^2)}{x^6} \\ &= \frac{(9-x^2)}{x^4} = \frac{(3-x)(3+x)}{x^4} \\ \Longrightarrow &\quad \text{critical point: } x = \pm 3. \end{aligned}$$

The interval of increasing/decreasing of f from the sign line of f':

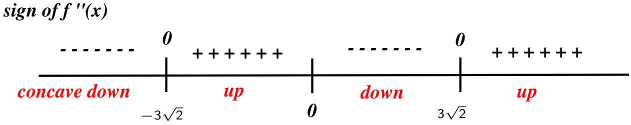


Compute the 2-st derivatives and their sign lines:

$$f''(x) = \frac{x^4(-2x) - (9 - x^2)(4x^3)}{x^8} = \frac{2(x^2 - 18)}{x^5} = \frac{2(x - 3\sqrt{2})(x + 3\sqrt{2})}{x^5}$$

$\Longrightarrow$ Inflection points: $(3\sqrt{2}, f(3\sqrt{2}))$ and $(-3\sqrt{2}, f(-3\sqrt{2}))$.

The concavity of f can be found easily from the sign line of f'':



Graph Sketching: Example

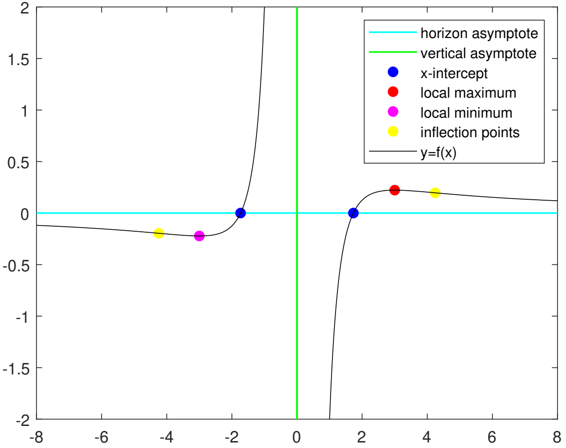
Putting together all the information above

- vertical asymptote: y-axis
- horizontal asymptote: x-axis
- x-intercept: $x = \pm\sqrt{3}$
- local minimum: $f(-3) = -\dfrac{2}{9}$
- local maximum: $f(3) = \dfrac{2}{9}$
- inflection points: $\left(-3\sqrt{2}, -\dfrac{5\sqrt{2}}{36}\right)$, $\left(3\sqrt{2}, \dfrac{5\sqrt{2}}{36}\right)$

and plotting a suitable number of points, we can sketch the graph.

Graph Sketching: Example

$$y = f(x) = \frac{x^2 - 2}{x^3}$$



Graph Sketching by MATLAB

```
% crate the figure
figure;

% enumerate point: -8, -7.99 .... 7.99, 8
x = -8: 0.01: 8;

% compute f(x): f(-8), f(-7.99) .... f(7.99), f(8)
y = (x.^2 - 3)./x.^3;

% plot graph by connecting
% (-8, f(-8)), (-7.99, f(-7.99)) ... (8, f(8)), (-8, f(-8))
plot(x, y, 'k-');  hold on;

% set the range of display
xlim([-8, 8]); ylim([-2, 2]);

% draw x-axis and y-axis
plot(xlim, [0,0], 'b', 'LineWidth', 1); hold on;
plot([0,0], ylim, 'r', 'LineWidth', 1); hold on;

% add the legends
legend('y=f(x)', 'x-axis', 'y-axis')
```