Calculus IB: Lecture 12

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Outline

1. Extreme Values of Functions
2. The Mean Value Theorem
3. Using 1st and 2nd Derivatives in Graphing

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2. The Mean Value Theorem
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Extreme Values of a Function

When studying a function, we sometimes need to determine its largest function value or smallest function value.

Suppose we have a function f, and c is a real number in its domain D.

- $f(c)$ is called the *global maximum* (or *absolute maximum*) of f on D if $f(c) \geq f(x)$ for *all* real number x in D.
- $f(c)$ is called the *global minimum* (or *absolute minimum*) of f on D if $f(c) \leq f(x)$ for *all* real numbers x in D.
- $f(c)$ is called a *local maximum* (or *relative maximum*) of f on D if $f(c) \geq f(x)$ for *numbers x in D which are "near" c*.
- $f(c)$ is called a *local minimum* (or *relative minimum*) of f on D if $f(c) \leq f(x)$ for *numbers x in D which are "near" c*.
- An *extremum* (or extreme value) is either a maximum or minimum, absolute or local.

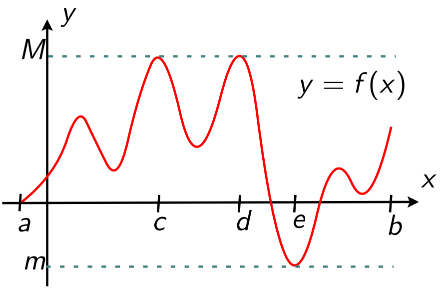
Precise definition of local maximum: Suppose we have a function f, and c is a number in its domain D. $f(c)$ is called a *local maximum* (or *relative maximum*) of f on D if there exists an $\delta > 0$ such we have $f(c) \geq f(x)$ whenever $|x - c| < \delta$ holds for any x in domain D.

Precise definition of local minimum: Suppose we have a function f, and c is a number in its domain D. $f(c)$ is called a *local minimum* (or *relative minimum*) of f on D if there exists an $\delta > 0$ such we have $f(c) \leq f(x)$ whenever $|x - c| < \delta$ holds for any x in domain D.

Note also that any global maximum (global minimum respectively) is automatically a local maximum (local minimum respectively).

Extreme Values of Functions

Here is a simple graph of $y = f(x)$ defined on a closed interval $[a, b]$ to illustrate the usage of these terms above:



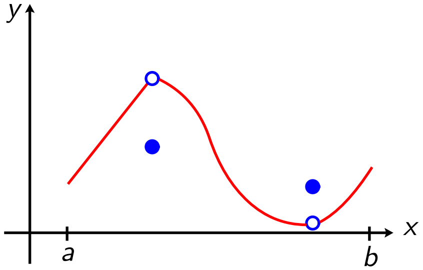
The global maximum is reached at two points, $M = f(c) = f(d)$, and the global minimum is reached at one point, $m = f(e)$.

In addition, there are three other local maximum, and four other local minimum.

A function may not have maximum or minimum though. Just look at the graph below:

You may run along the graph to get closer and closer to the two "holes" (y-values), but can never reach those heights as any function value.

This is obviously due to the existence of points of discontinuity.



We are more interested in extreme values of continuous functions.

The Extreme Value Theorem

Theorem (Extreme Value Theorem)

If f is continuous on a closed interval, then f attains an global maximum $f(c)$ and an global minimum $f(d)$ at some numbers c and d in $[a, b]$.

The global maximum/minimum may be reached at the boundary points of the closed interval $[a, b]$, or at points inside the open interval (a, b).

If $f(c)$ is a local maximum for some c in (a, b), then $f(c+h) \leq f(c)$ and $f(c-h) \leq f(c)$ for all sufficiently small $h > 0$. If $f'(c)$ exists, we have

$$0 \leq \lim_{h \to 0^+} \frac{f(c-h) - f(c)}{(c-h) - c} = f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{(c+h) - c} \leq 0,$$

i.e., $f'(c) = 0$.

A number c in the domain of f is called a *critical number* or *critical point* if either $f'(c) = 0$ or $f'(c)$ does not exist.

Finding Global Maximum/Minimum

Theorem (Fermat's Theorem)

If f has a local maximum or local minimum at an interior point c, and if $f'(c)$ exists, then $f'(c) = 0$.

As a result, we obtain a basic approach to find the global maximum and minimum of a differentiable function f on a closed interval $[a, b]$ is:

1. Find all critical points of f in (a, b), and the respective function values.
2. Find the function values of f at the boundary points of the interval $[a, b]$.
3. Just compare these function values above to find the largest (global maximum) and smallest (global minimum).

Example

Find global maximum/minimum of function $f(x) = x^3$ on interval $[-2, 3]$.

We can verify 0 is unique critical point of f since

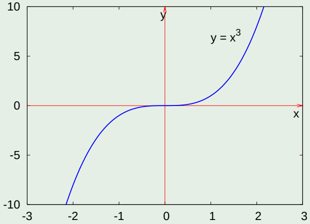
$$f'(x) = 3x^2 = 0 \iff x = 0.$$

A straightforward comparison of function values leads to

$$f(-2) = -8,$$

$$f(0) = 0,$$

$$f(3) = 27.$$



Hence $f(-2) = -8$ global minimum and $f(3) = 27$ is global maximum, but $f(0) = 0$ is neither local maximum value nor local minimum value.

Finding Global Maximum/Minimum

Example

Find the absolute maximum and minimum values of $f(x) = 3x(4 - \ln x)$ on the interval $\left[1, e^5\right]$.

We desire to find the critical point of f by its derivative:

$$f'(x) = 3(4 - \ln x) + 3x(-\frac{1}{x}) = 9 - 3\ln x \qquad \text{(Product Rule)}$$

$$f'(x) = 0 \iff \ln x = 3 \iff x = e^3$$

At the only critical point $x = e^3$: $f(e^3) = 3e^3(4 - \ln e^3) = 3e^3$.

At the endpoints of the interval $[1, e^5]$:

$$f(1) = 3(4 - \ln 1) = 12, \qquad f(e^5) = 3e^5(4 - \ln e^5) = -3e^5.$$

Then, the global maximum is $3e^3$, and the global minimum is $-3e^5$.

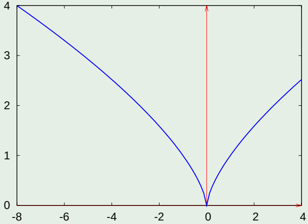
Example

Find the absolute maximum and minimum values of $f(x) = x^{\frac{2}{3}}$ on the interval $[-8, 4]$.

The derivative of f is

$$f'(x) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3x^{-\frac{1}{3}}},$$

which can never be 0.



However, $x = 0$ is a critical point since $f'(0)$ does not exist. Hence, this point also should be taken into consideration.

Compare function values at the critical point and endpoints:

$$f(0) = 0, \qquad f(-8) = (-8)^{\frac{2}{3}} = 4, \qquad f(4) = 4^{\frac{2}{3}} = \sqrt[3]{16}.$$

The absolute maximum value is 4, and the absolute minimum value is 0.

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1 Extreme Values of Functions

2 The Mean Value Theorem

3 Using 1st and 2nd Derivatives in Graphing

Rolle's Theorem

Combining the extreme value theorem and Fermat's theorem, it is easy to conclude Rolle's theorem.

Theorem (Extreme Value Theorem)

If f is continuous on a closed interval, then f attains an global maximum $f(c)$ and an global minimum $f(d)$ at some numbers c and d in $[a, b]$.

Theorem (Fermat's Theorem)

If f has a local maximum or local minimum at an interior point c, and if $f'(c)$ exists, then $f'(c) = 0$.

Theorem (Rolle's Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and $f(a) = f(b)$ and $a < b$, then $f'(c) = 0$ for some number $c \in (a, b)$.

Rolle's Theorem

Theorem (Rolle's Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), $f(a) = f(b)$ and $a < b$, then $f'(c) = 0$ for some number $c \in (a, b)$.

Proof.

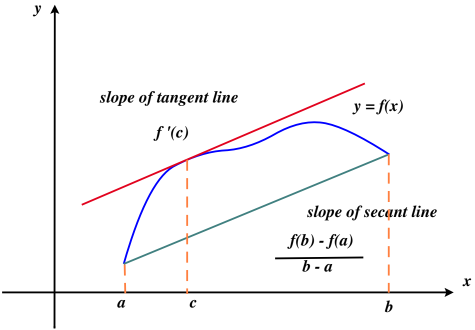
The idea is pretty simple: either $f(x) = f(a) = f(b)$ for all x in $[a, b]$, and thus $f'(x) = 0$ for all x in (a, b), or a maximum is reached at some c in (a, b) by the extreme value theorem so that $f'(c) = 0$ by the Fermat's theorem. □

Theorem (Mean Value Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

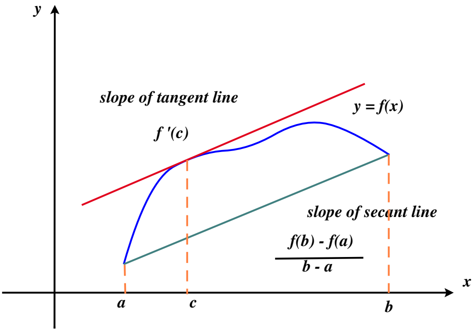
for some $c \in (a, b)$, or equivalently $f(b) - f(a) = f'(c)(b - a)$.



Mean Value Theorem

The proof of mean value theorem is based on the gap between the graph of f and the secant line joining the endpoints of the graph

$$h(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right]$$



Mean Value Theorem

The proof of mean value theorem is based on the gap between the graph of f and the secant line joining the endpoints of the graph

$$h(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right]$$

It is easy to verify $h(a) = h(b) = 0$, then $h(x)$ satisfies the conditions of Rolle's theorem. Hence, there exists some c in (a, b) such that $h'(c) = 0$:

$$h'(c) = f'(c) - \left[0 + \frac{f(b) - f(a)}{b - a}\right] = 0$$

$$\Longrightarrow f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{or} \quad f(b) - f(a) = f'(c)(b - a)$$

Here are some consequences of the mean value theorem:

- If $f' = 0$ on the whole interval (a, b), then f is a constant function on the interval. (for any $a < x_1 < x_2 < b$, we have $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0$, i.e., $f(x_1) = f(x_2)$)
- If $f'(x) > 0$ for all x in an interval (a, b), then $f(x)$ is an increasing function on (a, b). (for any $a < x_1 < x_2 < b$, we have $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$, for some c between x_1 and x_2, i.e., $f(x_2) > f(x_1)$ since $f'(c) > 0$)
- If $f'(x) < 0$ for all x in an interval (a, b), then $f(x)$ is an decreasing function on (a, b).

Another Mean Value Theorem

Exercise

If f is twice differentiable on (a, b) and continuous on $[a, b]$, then

$$f(b) - f(a) - f'(a)(b-a) = \frac{f''(c)}{2}(b-a)^2$$

for some $c \in (a, b)$.

Hint: Consider

$$h(x) = f(x) - f(a) - f'(a)(x-a) - \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}(x-a)^2$$

$$h'(x) = f'(x) - f'(a) - 2 \cdot \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}(x-a)$$

What if you now apply the Rolle's Theorem? What is $h''(x)$?

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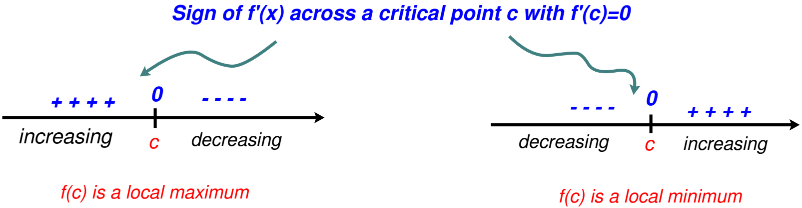
Using 1st and 2nd Derivatives in Graphing

A lot about function $y = f(x)$ can be found by $f'(x)$ and $f''(x)$.

- $f(x)$
  - $f'(x)$
    - Critical Points : $f'(x) = 0$ (or undefined)
    - Intervals of Increase/Decrease ($\pm$ sign of $f'(x)$)
    - 1st Derivative Test for Local Extrema (i.e., look at the sign line of f'.)
  - $f''(x)$
    - Convex/Concave Intervals ($\pm$sign of $f''(x)$)
    - Inflection Points - where concavity changes.
    - 2nd Derivative Test for Local Extrema

First Derivative Test



$f(c)$ is neither a local maximum nor a local minimum if the sign of f' does not change across c.

Second order derivative test

$$f'(c) = 0 \text{ and } \begin{cases} f''(c) < 0, & \text{then } f(c) \text{ is a local maximum.} \\ f''(c) > 0, & \text{then } f(c) \text{ is a local minimum.} \end{cases}$$

This follows from the limit definition of derivative and the 1st derivative test. If $f'(c) = 0$ and $f''(c) > 0$, then we have

$$\lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h \to 0} \frac{f'(c+h) - 0}{h} = f''(c) > 0$$

which means when h is very close to 0, $\dfrac{f'(c+h)}{h} \approx f''(c) > \dfrac{1}{2} f''(c) > 0$.

Hence if $h > 0$, we have $f'(c+h) > \frac{h}{2} f''(c) > 0$; and if $h < 0$, we have $f'(c+h) < \frac{h}{2} f''(c) < 0$. Thus $f(c)$ is a local minimum by the 1st derivative test.

Second Derivative Test

How to obtain $\frac{f'(c+h)}{h} \approx f''(c) > \frac{1}{2}f''(c) > 0$ from the limit?

The formal description: there exists $\delta > 0$ such that for any $0 < |h| < \delta$, we have

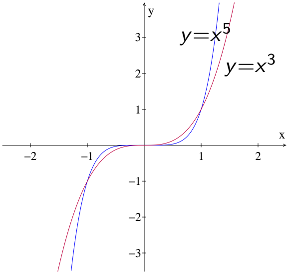
$$\frac{f'(c+h)}{h} > 0.$$

Try to prove it by (ε, δ)-definition of limit and local minimum/maximum.

Second Derivative Test

Please note that

- $f'(c) = 0$ and $f''(c) \geq 0$ does not mean $f(c)$ is a local minimum
- $f'(c) = 0$ and $f''(c) \leq 0$ does not mean $f(c)$ is a local maximum



$f(0)$ is neither local minimum nor local maximum

Second Derivative Test

Suppose f, f' and f'' are well defined on (a, b) and c in (a, b). Note that sufficient condition and necessary condition of local extrema are different.

- $f'(c) = 0$ and $f''(c) > 0$ mean $f(c)$ is a local minimum
- $f(c)$ is a local minimum means $f'(c) = 0$ and $f''(c) \geq 0$
- $f'(c) = 0$ and $f''(c) < 0$ mean $f(c)$ is a local maximum
- $f(c)$ is a local maximum means $f'(c) = 0$ and $f''(c) \leq 0$

Second Derivative Test

Exercise

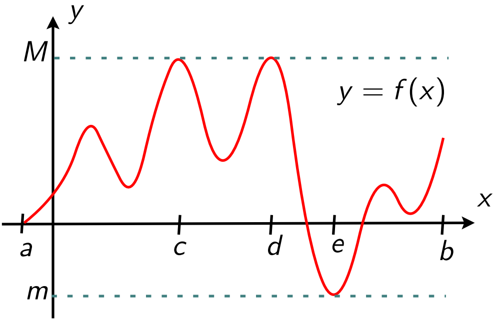
Under conditions in last page, prove $f(c)$ is a local minimum means $f'(c) = 0$ and $f''(c) \geq 0$.

Hint: If $f(c)$ is a local minimum for c in (a, b), then first derivative test means $f'(c) = 0$. Hence, we only need to show $f''(c) \geq 0$. We suppose $f''(c) < 0$, then the exercise on page 18 will lead to contradiction.

The result $f''(c) \geq 0$ cannot be improved to $f''(c) > 0$. We just need consider constant function $f(x) = c$. Every $f(x)$ are local minimum or maximum and $f'(x) = 0$, but $f''(0) = 0$ is not strictly greater than 0.

Endpoints and Local Minimum/Maximum

If $f(x)$ is differentiable on (a, b) and continuous on $[a, b]$, can we conclude the endpoints $f(a)$ (or $f(b)$) must be a local minimum or local maximum?



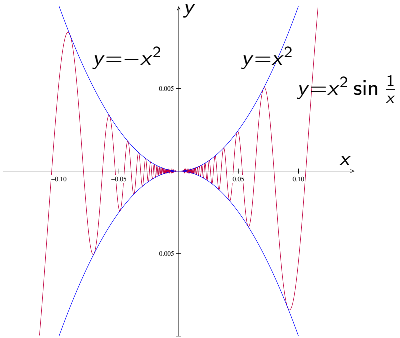
It is incorrect!

Endpoints and Local Minimum/Maximum

Consider the example we have mentioned in section chain rule:

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Here we restrict the domain as $[0, 1]$. What about $x = 0$?



Note that $g(0)$ is neither local minimum or local maximum of

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Examples: $f(x) = x^3 - 2x^2 + x - 1$

Find the intervals of increase/decrease, and local extrema of the function $f(x) = x^3 - 2x^2 + x - 1$.

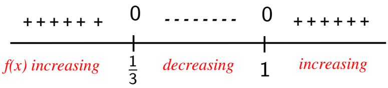
By differentiating the function, we have

$$f'(x) = 3x^2 - 4x + 1 = (3x-1)(x-1).$$

The critical points can be found by solving the equation $f'(x) = 0$:

$$(3x-1)(x-1) = 0 \iff x = \frac{1}{3}, \text{ or } x = 1.$$

Let's check the sign of $f'(x)$ in intervals $x < \frac{1}{3}$, $\frac{1}{3} < x < 1$, and $x > 1$:



By the first derivative test, we have

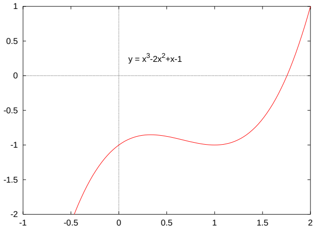
$$f\left(\frac{1}{3}\right) = -\frac{23}{27} \text{ is a local maximum and } f(1) = -1 \text{ is a local minimum.}$$

Examples: $f(x) = x^3 - 2x^2 + x - 1$

If you prefer using a table instead of a sign line:

interval	$x < \frac{1}{3}$	$x = 0$	$0 < x < 1$	$x = 1$	$1 < x$
$f'(x)$	$+ve$	0	$-ve$	0	$+ve$
$f(x)$	increasing	local max	decreasing	local min	increasing

The graph of $y = x^3 - 2x^2 + x - 1$:



No global extrema:

$$\lim_{x \to +\infty} f(x) = +\infty$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

Examples: $f(x) = (x^2 - 3)e^{-x}$

Find the intervals of increase/decrease, and local extrema of the function $f(x) = (x^2 - 3)e^{-x}$.

We have

$$\begin{aligned} f'(x) =& e^{-x}\frac{d(x^2-3)}{dx} + (x^2-3)\frac{de^{-x}}{dx} \\ =& 2xe^{-x} - (x^2-3)e^{-x} = -(x^2-2x-3)e^{-x}, \end{aligned}$$

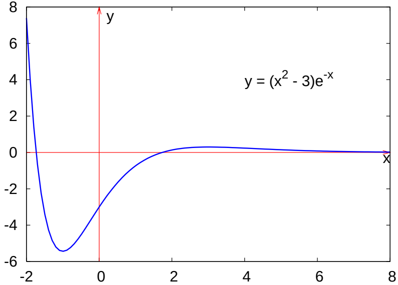
then the critical points are:

$$-(x^2-2x-3)e^{-x} = 0 \iff (x-3)(x+1) = 0 \iff x = -1 \text{ or } x = 3$$

interval	$x < -1$	$x = -1$	$-1 < x < 3$	$x = 3$	$3 < x$
$f'(x)$	$-ve$	0	$+ve$	0	$-ve$
$f(x)$	decreasing	local min	increasing	local max	decreasing

local minimum: $f(-1) = -2e$ local maximum: $f(3) = 6e^{-3}$

Graph of $f(x) = (x^2 - 3)e^{-x}$:



Note that the graph suggests $\lim_{x \to +\infty} (x^2 - 3)e^{-x} = 0$. We will discuss this type of limits in more details when discussing l'Hôpital's Rule later.

The local minimum $f(-1) = -2e$ is actually an global minimum.