Calculus IB: Lecture 11

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Outline

1. Rates of Changes

2. Higher Order Derivatives

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2. Higher Order Derivatives

Rates of Change

When a function $y = f(x)$ describes the relation between two quantities represented by x and y respectively, the derivative function

$$f'(x) \quad \text{or} \quad \frac{dy}{dx}$$

is considers as the *rate of change* of the quantity y with respect to the quantity x.

The main idea about related rates is essentially the following.

Given some quantities

$$\left.\begin{array}{l} q_1 = q_1(t) \\ q_2 = q_2(t) \\ \vdots \\ q_n = q_n(t) \end{array}\right\} \quad \begin{array}{c}\text{which are all functions of } t, \\ \text{where } t \text{ may represent time or some other quantity,}\end{array}$$

if there is an equation relating all these quantities, then

$$\frac{d}{dt} \text{ of both sides of the relation}$$

$$\xrightarrow{gives} \quad \text{an equation relating the rates of changes } \frac{dq_1}{dt}, \frac{dq_2}{dt}, \ldots, \frac{dq_n}{dt}$$

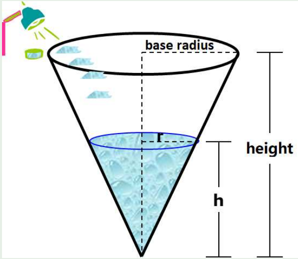
Examples of Related Rates

Example (cone container)

Water is flowing into a cone container at a rate of $2\mathrm{m}^3/\mathrm{min}$. Suppose the shape of the cone container satisfies

$$\frac{\text{base radius}}{\text{height}} = \frac{r}{h} = \frac{1}{2}.$$

How fast is the water level rising (speed of h increasing) when the water in the container is 3m in depth?



Example (cone container)

At time t, volume of water with depth h is $V = \frac{1}{3}\pi r^2 h$ and $r = \frac{h}{2}$.

We also know that $\frac{dV}{dt} = 2\mathrm{m}^3/\mathrm{min}$. The question is find $\left.\frac{dh}{dt}\right|_{h=3}$.

Here the equation relating V and h, both are functions of t, is

$$V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Taking the derivatives of both sides with respect to t,the chain rule means

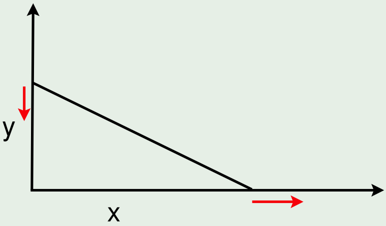
$$\frac{dV}{dt} = \frac{\pi}{4}h^2 \cdot \frac{dh}{dt}$$

When $h = 3$m, we have

$$2 = \frac{\pi}{4}(3)^2 \cdot \left.\frac{dh}{dt}\right|_{h=3} \quad \Longrightarrow \quad \left.\frac{dh}{dt}\right|_{h=3} = \frac{8}{9\pi}\ \mathrm{m}/\mathrm{min}$$

Example (ladder and wall)

A ladder of 5m long is leaning again the wall. If the bottom of the ladder is pulled away from the wall at a rate of 1m/min, how fast is the top of the ladder dropping when the bottom of the ladder is 3m away from the wall?



Example (ladder and wall)

At time t, we have:

- distance between the wall and the bottom of the ladder: $x = x(t)$
- distance between the ground and the top of the ladder: $y = y(t)$
- known condition: $\frac{dx}{dt} = 1\mathrm{m}/\mathrm{min}$

We want to find $\left.\frac{dy}{dt}\right|_{x=3}$ and the relation between x and y is $x^2 + y^2 = 5^2$.

Differentiating both sides with respect to t, we have

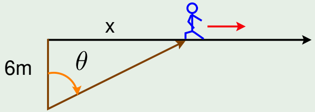
$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = \frac{d}{dt} \cdot (25) = 0.$$

When $x = 3$m, $y = \sqrt{5^2 - 3^2} = 4$m, and hence

$$2 \cdot 3 \cdot 1 + 2 \cdot 4 \cdot \left.\frac{dy}{dt}\right|_{x=3} = 0 \quad \Longrightarrow \quad \left.\frac{dy}{dt}\right|_{x=3} = -\frac{3}{4} \quad (\mathrm{m}/\mathrm{min})$$

Example (walking man)

A man starts walking along a straight line at a velocity of 1.5m/s. A light beam 6m from the road is tracking the man. How fast is the angle of the beam turning when the man is 9m away from the starting point?



At time t, we have:

- distance travelled by the man : $x = x(t)$
- turning angle of the light beam: $\theta = \theta(t)$
- known condition: $\dfrac{dx}{dt} = 1.5 \text{ m/s}$

We want to find $\left.\dfrac{d\theta}{dt}\right|_{x=9}$.

Example (walking man)

The relation between x and θ is: $\tan\theta = \frac{x}{6}$.

Differentiating both sides with respect to t, we have

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{6}\frac{dx}{dt}$$

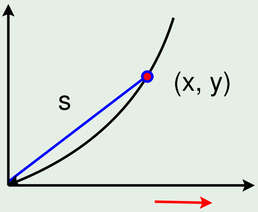
When $x = 9$m, $\sec\theta = \frac{\sqrt{9^2+6^2}}{6} = \frac{\sqrt{117}}{6}$m, and hence

$$\frac{117}{36}\cdot\left.\frac{d\theta}{dt}\right|_{x=9} = \frac{1}{6}\cdot 1.5 \quad\Longrightarrow\quad \left.\frac{d\theta}{dt}\right|_{x=9} = \frac{9}{117}\ (\mathrm{rad/s}).$$

Examples of Related Rates

Example (travelling along the curve)

A particle is travelling along the graph of the function $y = x^2$ such that the velocity of the particle in x-direction is 1m/s. How fast is the distance between the particle and the origin increasing when $x = 2$m?



At time t, we have:

- x coordinate of the particle: $x = x(t)$
- y coordinate of the particle: $y = y(t)$
- distance between the particle and the origin: $s = s(t)$
- known condition: $\frac{dx}{dt} = 1 \text{ m/s}$

Example (travelling along the curve)

We want to find $\left.\frac{ds}{dt}\right|_{x=2}$. Relations of the quantities:

$$\begin{cases} y = x^2 \\ s^2 = x^2 + y^2 \end{cases} \xrightarrow{d/dt} \begin{cases} \dfrac{dy}{dt} = 2x\dfrac{dx}{dt} \\ 2s\dfrac{ds}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} \end{cases}$$

When $x = 2$m, $y = 4$m, $s = \sqrt{2^2 + 4^2} = \sqrt{20}$m and

$$\begin{cases} \left.\dfrac{dy}{dt}\right|_{x=2} = 2 \cdot 2 \cdot 1 = 4 \\ 2\sqrt{20} \cdot \left.\dfrac{ds}{dt}\right|_{x=2} = 2 \cdot 2 \cdot 1 + 2 \cdot 4 \cdot \left.\dfrac{dy}{dt}\right|_{x=2} = 4 + 8 \cdot 4 \end{cases}$$

Hence $\left.\frac{ds}{dt}\right|_{x=2} = \frac{18}{\sqrt{20}}$ m/s.

Outline

1 Rates of Changes

2 Higher Order Derivatives

Second Order Derivative (or Second Derivative)

If $s = s(t)$ is the position function of a particle moving along a line represented by the x axis, then

$$\frac{dx}{dt} = \text{ velocity function } = v(t)$$

$$\frac{dv}{dt} = \text{ acceleration function} = a(t)$$

In particular, if m is the mass of the particle, and F is the force acting on the particle, Newton's Second Law $F = ma$ can be expressed as

$$F = m\frac{dv}{dt} = m\frac{d^2s}{dt^2}$$

where the second derivative means "the derivative of the derivative":

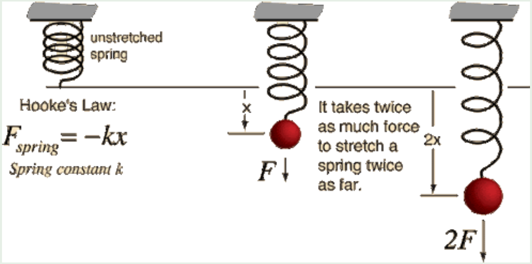
$$s''(t) = \frac{d^2s}{dt^2} \overset{means}{=} \frac{d}{dt}\left(\frac{ds}{dt}\right).$$

Example

The Hooke's Law for spring-mass system, which says that the spring force acting on a mass attached to the spring is proportional to the displacement of the mass from its equilibrium position, can be expressed as

$$m\frac{d^2s}{dt^2} = -kx$$

where k is the spring constant, and x is the displacement of the mass from its equilibrium position (spring is unstretched).



Higher Order Derivatives

The second order derivative of f is the derivative of the derivative of f:

$$\frac{d^2 f(x)}{dx^2} = f''(x) = (f')'(x).$$

The third order derivative of f is the derivative of the second order derivative of f:

$$\frac{d^3 f(x)}{dx^3} = f'''(x) = (f'')'(x).$$

In general, the n-th order derivative of f is the derivative of the $(n-1)$-th order derivative of f:

$$\frac{d^n f(x)}{dx^n} = f^{(n)}(x) = \left(f^{(n-1)}\right)'(x).$$

Higher Order Derivatives

Example

Calculate the second order derivative of $f(x) = \sin x^2$.

We apply the chain rule to calculate the first derivative:

$$f'(x) = (\cos x^2) \cdot 2x = 2x \cos x^2.$$

Then we need to use product rule and the chain rule to calculate the second derivative:

$$\begin{aligned} f''(x) &= \left(f'(x)\right)' \\ &= \left(2x \cos x^2\right)' \\ &= (2x)' \cdot \cos x^2 + 2x \cdot \left(\cos x^2\right)' \\ &= 2 \cos x^2 - 4x^2 \sin x^2. \end{aligned}$$

Higher Order Derivatives

Example

For any polynomial $P_n(x)$ of degree n, show that $P_n^{(k)}(x) = 0$ for any integer $k > n$.

Since $\frac{d}{dx}(x^m)' = mx^{m-1}$, we have $\frac{d^k}{dx^k}(x^m) = 0$ if $k > m$.

Hence, if $k > n$, we have

$$
\begin{aligned}
&\frac{d^k}{dx^k} P_n(x) \\
=&\frac{d^k}{dx^k}\left(a_0 + a_1 x + \cdots + a_n x^n\right) \\
=&a_0 \cdot \frac{d^k}{dx^k}(1) + a_1 \cdot \frac{d^k}{dx^k}(x) + \cdots + a_n \cdot \frac{d^k}{dx^k}(x^n) \\
=&0 + 0 + \cdots + 0 = 0.
\end{aligned}
$$

Exercise

For any polynomial $P_n(x)$ of degree n and positive integer $k \leq n$, find $P_n^{(k)}(x)$.

Exercise

Let the function $y = f(x)$ defined implicitly by equation $x^2 + y^2 = 1$. Use implicit differentiation to show that

$$\frac{d^2y}{dx^2} = -\frac{1}{y^3}.$$