Calculus IB: Lecture 10

Luo Luo

Department of Mathematics, HKUST

http://luoluo.people.ust.hk/

Outline

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Derivatives of Trigonometric Functions

1. $$\frac{d \sin x}{dx} = \cos x$$
2. $$\frac{d \cos x}{dx} = -\sin x$$
3. $$\frac{d \tan x}{dx} = \sec^2 x$$
4. $$\frac{d \cot x}{dx} = -\csc^2 x$$
5. $$\frac{d \sec x}{dx} = \sec x \tan x$$
6. $$\frac{d \csc x}{dx} = -\csc x \cot x$$

Derivatives of $\sin x$

Using the identities (Lecture-L04)

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

with $\alpha = x + h$ and $\beta = x$, we have

$$\begin{aligned}
\frac{d\sin x}{dx} &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} \\
&= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\cdot\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \\
&= \cos\left(\lim_{h\to 0}\left(x+\frac{h}{2}\right)\right)\cdot\lim_{t\to 0}\frac{\sin t}{t} \qquad \text{let } t = \frac{h}{2} \\
&= \cos x\cdot 1 = \cos x.
\end{aligned}$$

Derivatives of $\cos x$

Using the identities (Lecture-L04)

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

with $\alpha = x + h$ and $\beta = x$, we have

$$\begin{aligned}
\frac{d\cos x}{dx} &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{-2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} \\
&= -\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right)\cdot\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \\
&= -\sin\left(\lim_{h\to 0}\left(x+\frac{h}{2}\right)\right)\cdot\lim_{t\to 0}\frac{\sin t}{t} \qquad \text{let } t = \frac{h}{2} \\
&= -\sin x\cdot 1 = -\sin x.
\end{aligned}$$

Derivatives of $\tan x$

By using $\dfrac{d \sin x}{dx} = \cos x$, $\dfrac{d \cos x}{dx} = -\sin x$ and quotient rule we have

$$\begin{aligned}
\frac{d \tan x}{dx} &= \frac{d}{dx}\frac{\sin x}{\cos x} \\
&= \frac{\cos x \dfrac{d \sin x}{dx} - \sin x \dfrac{d \cos x}{dx}}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x.
\end{aligned}$$

More Examples

Example

Find the derivative of $\dfrac{e^x \sin x}{x^2}$.

$$\frac{d}{dx}\frac{e^x \sin x}{x^2} = \frac{x^2 \dfrac{de^x \sin x}{dx} - e^x \sin x \dfrac{dx^2}{dx}}{x^4} \qquad \text{(Quotient Rule)}$$

$$= \frac{x^2 \left(e^x \dfrac{d \sin x}{dx} + \sin x \dfrac{de^x}{dx} \right) - 2xe^x \sin x}{x^4} \qquad \text{(Product Rule)}$$

$$= \frac{x^2 (e^x \cos x + e^x \sin x) - 2xe^x \sin x}{x^4}$$

$$= \frac{e^x (x \cos x + x \sin x - 2 \sin x)}{x^3}$$

Exercise of Derivatives

Exercise

Show that

1. $$\frac{d\cot x}{dx} = -\csc^2 x,$$
2. $$\frac{d\sec x}{dx} = \sec x\tan x,$$
3. $$\frac{d\csc x}{dx} = -\csc x\cot x.$$

More Examples

Example

Differentiate $y = (3x^4 - 2x^2 + 1)^3$.

Let $u = 3x^4 - 2x^2 + 1$, then $y = u^3$, $\dfrac{dy}{du} = 3u^2$, and

$$\frac{du}{dx} = 3 \cdot 4x^{4-1} - 2 \cdot 2x^{2-1} + 0 = 12x^3 - 4x.$$

Hence by the chain rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= 3u^2 \cdot (12x^3 - 4x) \\ &= 3(3x^4 - 2x^2 + 1)^2(12x^4 - 4x).\end{aligned}$$

More Examples

Example

Differentiate $y = \left(\frac{x}{1+x}\right)^{\frac{1}{3}}$.

Let $u = \frac{x}{1+x}$, then $y = \sqrt[3]{u} = u^{\frac{1}{3}}$. We have

$$\frac{dy}{du} = \frac{1}{3}u^{\frac{1}{3}-1} = \frac{1}{3}u^{-\frac{2}{3}},$$

$$\frac{du}{dx} = \frac{(1+x)\cdot\frac{dx}{dx} - x\frac{d(1+x)}{dx}}{(1+x)^2} = \frac{1}{(1+x)^2}.$$

By the chain rule, we obtain

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{3}\left(\frac{x}{1+x}\right)^{-\frac{2}{3}}\frac{1}{(1+x)^2} = \frac{1}{3}x^{-\frac{2}{3}}(1+x)^{-\frac{4}{3}}$$

More Examples

Exercise

Define $y = (1+x^2)\sin(2x^2+e^{x^2}) + e^{\sin\ln x}$. Show that

$$\frac{dy}{dx} = 2x\sin(2x^2+e^{x^2}) + (1+x^2)\cos(2x^2+e^{x^2})(4x+2xe^{x^2}) + \frac{1}{x}e^{\sin\ln x}\cos\ln x.$$

Outline

Derivatives of Inverse Functions

Theorem (Derivatives of Inverse Function)

Suppose f is a differentiable and has inverse function f^{-1} over an interval I and x is a point in I such that $x = f(a)$ and $f'(a) \neq 0$, then f^{-1} is differentiable at x and its derivative is

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(x))}.$$

Proof.

The definition of inverse function means $f(f^{-1}(x)) = x$. We can regard $f(f^{-1}(x))$ as composition of f and f^{-1} and use chain rule to obtain

$$\left(f \circ f^{-1}\right)'(x) = 1 \Longrightarrow f'(f^{-1}(x)) \cdot \left(f^{-1}\right)'(x) = 1 \Longrightarrow \left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}$$

□

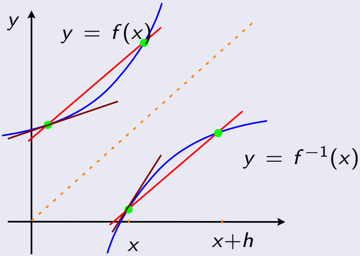
Derivatives of Inverse Functions

Exercise

By reflection across the line $y = x$, the tangent line to the graph of $y = f^{-1}(x)$ at the point $(x, f^{-1}(x))$ is reflected to the tangent line to the graph of $y = f(x)$ at $(f^{-1}(x), x)$. Try to explain

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}$$

in geometric view.



Example

Let $y = \ln x = f^{-1}(x)$ where $f(x) = e^x$.

Then $f'(x) = e^x$, and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

which means $\dfrac{d \ln x}{dx} = \dfrac{1}{x}$ we have proved before.

Derivatives of Inverse Functions

Example

If h is the inverse function of the increasing function $f(x) = x^3 + x + 1$, find $h'(1)$.

Note that we have

$$f'(x) = 3x^2 + 1.$$

Moreover, it is easy to verify $f(0) = 1$ which means $h(1) = 0$.

Hence, we have

$$h'(1) = \frac{1}{f'(h(1))} = \frac{1}{f'(0)} = \frac{1}{3 \cdot 0 + 1} = 1.$$

Derivatives of Inverse Functions

Example

Find the derivative of $\tan^{-1} x$.

Define $f(x) = \tan x$ with domain $-\pi/2 < x < \pi/2$. Then we have

$$f^{-1}(x) = \tan^{-1} x \quad \text{and} \quad f'(x) = \sec^2 x.$$

Hence, we have

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sec^2(\tan^{-1} x)} = \frac{1}{1 + \tan^2(\tan^{-1}(x))} = \frac{1}{1 + x^2},$$

where we use the identity (consider that $u = \tan^{-1} x$)

$$\frac{1}{\sec^2 u} = \frac{\cos^2 u}{\sin^2 u + \cos^2 u} = \frac{1}{\frac{\sin^2 u}{\cos^2 u} + 1} = \frac{1}{\tan^2 u + 1}$$

In other words, we have $\dfrac{d \tan^{-1} x}{dx} = \dfrac{1}{1 + x^2}$.

Outline

Implicit Differentiation

Sometimes, a function $y = f(x)$ can be defined implicitly by an equation of x and y of the form

$$F(x, y) = 0.$$

For example, the unit circle can be defined as

$$x^2 + y^2 = 1.$$

By solving the equation, we obtain two functions

$$y = \sqrt{1 - x^2} \quad \text{with} \quad y' = \frac{dy}{dx} = -\frac{x}{\sqrt{1 - x^2}}$$

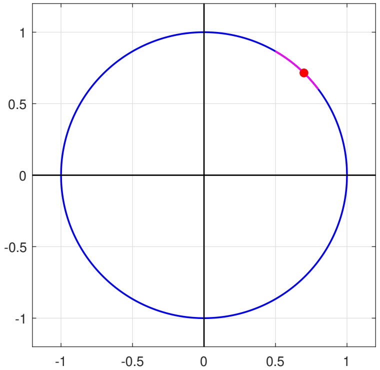
and

$$y = -\sqrt{1 - x^2} \quad \text{with} \quad y' = \frac{dy}{dx} = \frac{x}{\sqrt{1 - x^2}}.$$

The Implicit Function Theorem (beyond the requirement)

Graphically, we can observe $x^2 + y^2 = 1$ fails the familiar vertical line test, but it could pass the vertical line test locally.

Most of points on the circle we can choose a small neighborhood where our curve satisfies the vertical line test (determines y as a function of x).



The Implicit Function Theorem (beyond the requirement)

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Theorem (The Implicit Function Theorem)

Consider a continuously differentiable function $F(x, y)$ and a point (x_0, y_0) so that $F(x_0, y_0) = c$. If

$$\frac{\partial F}{\partial y}(x_0, y_0) = 0,$$

then there is a neighborhood of (x_0, y_0) so that whenever x is sufficiently close to x_0 there is a unique y so that $F(x, y) = c$. Moreover, this assignment is makes y a continuous function of x.

Implicit Differentiation

In general, it is difficult or impossible to find the explicit expression of $y = f(x)$ by $F(x, y)$, but we can express $y' = f'(x)$ by x and y.

We desire to find $f'(x)$ directly from the implicit form $F(x, y) = 0$ without solving $y = f(x)$.

Implicit differentiation can be done as follows:

$$F(x, y) = 0 \quad \xrightarrow{\frac{d}{dx} \text{ both sides}} \quad \text{an equation to solve for } \frac{dy}{dx}$$

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Implicit Differentiation

Example

Find the derivative of function $y = f(x)$ from the equation of unit circle.

We take the differentiation on both sides of the equation:

$$\begin{aligned} x^2 + y^2 = 1 \Longrightarrow & \frac{dx^2}{dx} + \frac{dy^2}{dx} = \frac{d1}{dx} \\ \Longrightarrow & 2x + \frac{dy^2}{dy} \cdot \frac{dy}{dx} = 0 \\ \Longrightarrow & 2x + 2y \cdot y' = 0 \\ \Longrightarrow & y' = -\frac{x}{y}. \end{aligned}$$

We can obtain a unique expression for the slope of the tangent line of unit circle at point (x, y) without the expression of $y = f(x)$.

Implicit Differentiation

Example

Find the derivative of function $y = f(x)$ from equation $\sin(xy) = x^2 + y$.

We take the differentiation on both sides of the equation:

$$\begin{aligned}
\sin(xy) = x^2 + y \Longrightarrow & \frac{d\sin(xy)}{dx} = \frac{dx^2}{dx} + \frac{dy}{dx} \\
\Longrightarrow & \cos(xy) \cdot \frac{dxy}{dx} = 2x + \frac{dy}{dx} \\
\Longrightarrow & \cos(xy) \cdot \left(x \cdot \frac{dy}{dx} + \frac{dx}{dx} \cdot y \right) = 2x + \frac{dy}{dx} \\
\Longrightarrow & y' = \frac{dy}{dx} = \frac{y\cos(xy) - 2x}{1 - x\cos(xy)}.
\end{aligned}$$

It is impossible to find explicit expression for $y = f(x)$ for this example.

Implicit Differentiation

Example

Find the derivative of $y = \sqrt[3]{2 - 2x^2}$.

The function can defined by equation $2x^2 + y^3 = 2$.

Just differentiate both sides as functions of x to get

$$\frac{d2x^2}{dx} + \frac{dy^3}{dx} = \frac{d2}{dx} \Longrightarrow 4x + 3y^2\frac{dy}{dx} = 0,$$

Then we have

$$\frac{dy}{dx} = -\frac{4x}{3y^2} = -\frac{4}{3}x(2 - 2x^2)^{-\frac{2}{3}}.$$

Derivative of Inverse Trigonometric Function

Example

Find derivatives of $y = \sin^{-1} x$.

We have $x = \sin y$ with $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and

$$\begin{aligned} x = \sin y \Longrightarrow & \frac{dx}{dx} = \frac{d \sin y}{dx} \\ \Longrightarrow & 1 = \frac{d \sin y}{dy} \cdot \frac{dy}{dx} \\ \Longrightarrow & 1 = \cos y \cdot \frac{dy}{dx} \end{aligned}$$

Since $\sin^2 y + \cos^2 y = 1$ and $\cos y \geq 0$ whenever $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, we have

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.$$

Derivative of Inverse Trigonometric Function

Exercise

Find derivatives of

$$\cos^{-1} x, \quad \cot^{-1} x, \quad \csc^{-1} x \quad \text{and} \quad \sec^{-1} x.$$

by implicit differentiation.

Chain Rule Version of Basic Derivative Formulas

The following chain rule versions of basic derivative formulas are convenient to use for calculation of derivatives.

$$\frac{d\blacksquare^p}{dx} = p\blacksquare^{p-1} \cdot \frac{d\blacksquare}{dx}$$

$$\frac{de^\blacksquare}{dx} = e^\blacksquare \cdot \frac{d\blacksquare}{dx}$$

$$\frac{d\ln\blacksquare}{dx} = \frac{1}{\blacksquare} \cdot \frac{d\blacksquare}{dx}$$

$$\frac{d\sin\blacksquare}{dx} = \cos\blacksquare \cdot \frac{d\blacksquare}{dx}$$

$$\frac{d\cos\blacksquare}{dx} = -\sin\blacksquare \cdot \frac{d\blacksquare}{dx}$$

$$\frac{d\tan\blacksquare}{dx} = \sec^2\blacksquare \cdot \frac{d\blacksquare}{dx}$$

$$\frac{d\sec\blacksquare}{dx} = \sec\blacksquare \cdot \tan\blacksquare \cdot \frac{d\blacksquare}{dx}$$

$$\frac{d\sin^{-1}\blacksquare}{dx} = \frac{1}{\sqrt{1-\blacksquare^2}} \cdot \frac{d\blacksquare}{dx}$$

$$\frac{d\tan^{-1}\blacksquare}{dx} = \frac{1}{1+\blacksquare^2} \cdot \frac{d\blacksquare}{dx}$$

Logarithmic Differentiation

Logarithmic differentiation is just a special case of implicit differentiation.

Example

Find the derivative of $y = \sqrt[3]{2-2x^2}$ by working with the equation

$$\ln y = \ln(2-2x^2)^{1/3} = \frac{1}{3}\ln(2-2x^2).$$

Just differentiating both sides as functions of x again:

$$\frac{d}{dx}\ln y = \frac{1}{3}\frac{d}{dx}\ln(2-2x^2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\cdot\frac{1}{2-2x^2}\cdot(-4x)$$

$$\frac{dy}{dx} = \frac{y}{3}\cdot\frac{-4x}{2-2x^2} = -\frac{4}{3}x(2-2x^2)^{-\frac{2}{3}}$$