Calculus IB: Lecture 09

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Outline

1. Basic Formulas of Derivatives
2. Rules of Differentiation
3. The Chain Rule

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Basic Formulas of Derivatives

Here are the derivatives of some elementary functions, which are the results of some limit computations.

1. $\dfrac{dc}{dx} = 0$, for any constant c
2. $\dfrac{dx^p}{dx} = px^{p-1}$, for any constant p
3. $\dfrac{de^x}{dx} = e^x, \qquad \dfrac{d\ln x}{dx} = \dfrac{1}{x}$
4. $\dfrac{d\sin x}{dx} = \cos x, \qquad \dfrac{d\cos x}{dx} = -\sin x$

In the case of p is a non-negative integer, we have

$$\begin{aligned}
\frac{dx^p}{dx} &= \lim_{h\to 0} \frac{(x+h)^p - x^p}{h} \\
&= \lim_{h\to 0} \frac{\left[(x+h)-x\right]\left[(x+h)^{p-1} + (x+h)^{n-2}x + \cdots + x^{p-1}\right]}{h} \\
&= \lim_{h\to 0} \frac{h\left[(x+h)^{p-1} + (x+h)^{n-2}x + \cdots + x^{p-1}\right]}{h} \\
&= \lim_{h\to 0} \left[(x+h)^{p-1} + (x+h)^{n-2}x + \cdots + x^{p-1}\right] \\
&= \lim_{h\to 0} \left[\underbrace{x^{p-1} + x^{p-2}x + \cdots + x^{p-1}}_{p \text{ terms}}\right] \\
&= px^{p-1}
\end{aligned}$$

What about p is not a non-negative integer?

Basic Formulas of Derivatives: $\frac{de^x}{dx} = e^x$

Recall that we have

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

By the limit definition of derivative,

$$\begin{aligned}\frac{de^x}{dx} &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h\to 0} e^x \cdot \frac{e^h - 1}{h} \\ &= e^x \lim_{h\to 0} \frac{e^h - 1}{h} \\ &= e^x\end{aligned}$$

Basic Formulas of Derivatives: $\frac{d \ln x}{dx} = \frac{1}{x}$

Recall that we have

$$\ln x = \log_e x$$

and

$$e = \lim_{h \to 0} (1 + h)^{\frac{1}{h}}.$$

Using the definition of derivative, we have

$$\begin{aligned}
\frac{d \ln x}{dx} &= \lim_{h \to 0} \frac{\ln(x + h) - \ln x}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \ln \left(\frac{x + h}{x} \right) \\
&= \lim_{h \to 0} \ln \left(1 + \frac{h}{x} \right)^{\frac{1}{h}}.
\end{aligned}$$

Let $y = \dfrac{h}{x}$ and $h \to 0$ means $y \to 0$, then

$$\begin{aligned}
\lim_{h\to 0} \ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}} &= \lim_{y\to 0} \ln\left(1+y\right)^{\frac{1}{xy}} \\
&= \lim_{y\to 0} \ln\left[\left(1+y\right)^{\frac{1}{y}}\right]^{\frac{1}{x}} \\
&= \lim_{y\to 0} \frac{1}{x} \ln\left(1+y\right)^{\frac{1}{y}} \\
&= \frac{1}{x} \lim_{y\to 0} \ln\left(1+y\right)^{\frac{1}{y}} \\
&= \frac{1}{x} \ln\left(\lim_{y\to 0} \left(1+y\right)^{\frac{1}{y}}\right)
\end{aligned}$$

The last step is based on the following theorem.

Theorem

Suppose function f satisfies $\lim_{y \to x_0} f(y) = u_0$ and function g is continuous at u_0, then the composition function $(g \circ f)(y) = g(f(y))$ holds that

$$\lim_{y \to y_0} (g \circ f)(y) = \lim_{u \to u_0} g(u) = g(u_0).$$

Let $g(u) = \ln u$ and $f(y) = (1 + y)^{\frac{1}{y}}$, we have

$$\lim_{y \to 0} \ln (1 + y)^{\frac{1}{y}} = \ln \left(\lim_{y \to 0} (1 + y)^{\frac{1}{y}} \right) = \ln e = 1,$$

since $g(u) = \ln u$ is continuous on its domain. Hence, we have $\dfrac{d \ln x}{dx} = \dfrac{1}{x}$.

Note that above theorem requires the continuity of g, rather than f.

Consider the following examples:

$$g(u) = \begin{cases} 2 & u = 2 \\ 1 & u \neq 2 \end{cases} \qquad \text{and} \qquad f(y) = 2.$$

Let $u_0 = y_0 = 2$. Then we have

$$\lim_{y \to y_0} (g \circ f)(y) = \lim_{y \to 2} (g \circ f)(y) = \lim_{y \to 2} g(f(y)) = \lim_{y \to 2} g(2) = 2.$$

On the other hand, we also have

$$u_0 = \lim_{y \to 2} f(y) = 2 \quad \text{and} \quad \lim_{u \to u_0} g(u) = \lim_{u \to 2} g(u) = 1 \neq \lim_{y \to y_0} (g \circ f)(y).$$

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3. The Chain Rule

Rules of Differentiation

Whenever f' and g' both exist, we have the following rules:

1. $\dfrac{d}{dx}(af + bg) = a\dfrac{df}{dx} + b\dfrac{dg}{dx} = af' + bg'$ for any constants a and b.
2. **Product Rule:** $\dfrac{d}{dx}(fg) = f\dfrac{dg}{dx} + g\dfrac{df}{dx} = fg' + gf'$
3. **Quotient Rule:** $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{g\,\frac{df}{dx} - f\,\frac{dg}{dx}}{g^2} = \dfrac{gf' - fg'}{g^2}$

Exercise

Prove the first rule $\dfrac{d}{dx}(af + bg) = a\dfrac{df}{dx} + b\dfrac{dg}{dx} = af' + bg'$.

The Proof of Product Rule

We prove this rule by the limit definition of derivative:

$$
\begin{aligned}
&(fg)'(x) \\
=&\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
=&\lim_{h\to 0}\frac{(f(x+h)g(x+h)-f(x+h)g(x))+(f(x+h)g(x)-f(x)g(x))}{h} \\
=&\lim_{h\to 0}\frac{f(x+h)(g(x+h)-g(x))}{h}+\lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))}{h} \\
=&\lim_{h\to 0}f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
=&f(x)g'(x)+g(x)f'(x)
\end{aligned}
$$

where we use $\lim_{h\to 0} f(x+h) = f(x)$ since a function is continuous at any point where f' exists.

Exercise

Prove the quotient rule: $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} = \frac{gf' - fg'}{g^2}$.

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The Chain Rule

Let F is compositions of two functions f and g:

$$F(x) = (f \circ g)(x) = f(g(x)),$$

such that

1. g is a differentiable at x (the derivative $g'(x)$ exists),
2. and f is a is differentiable at $g(x)$ (the derivative $f'(g(x))$ exists);

then $y = F(x) = (f \circ g)(x)$ is differentiable at x, and its derivative is

$$F'(x) = f'(g(x)) \cdot g'(x).$$

The Proof of Chain Rule

There is one idea for the proof

$$
\begin{aligned}
F'(x) &= \lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{h}\\
&= \lim_{h\to 0}\left[\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}\cdot\frac{g(x+h)-g(x)}{h}\right]\\
&= \underbrace{\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)}}_{\text{slope of } f \text{ at } g(x)}\cdot\underbrace{\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}}_{\text{slope of } g(x) \text{ at } x}\\
&= f'(g(x))\cdot g'(x)
\end{aligned}
$$

How to proof this equality?

The remain is to show that

$$\lim_{h\to 0}\frac{f(g(x+h))-f(g(x))}{g(x+h)-g(x)} = f'(g(x)).$$

The Proof of Chain Rule

Recall the (ε, δ)-definition of limit.

Definition

The expression $\lim_{h\to 0} w(x) = L$ means for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|w(h) - L| < \varepsilon$ whenever $0 < |h| < \delta$.

We must find $\delta > 0$ such that $w(h)$ is well defined whenever

$$0 < |h| < \delta.$$

However, the function

$$w(h) = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}$$

is not well defined when $g(x+h) - g(x) = 0$.

The Proof of Chain Rule

For constant function such that $g(x) = a$, it is obviously

$$w(h) = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}$$

is undefined.

In such case, we have

$$F(x) = f(g(x)) = f(a),$$

which is also a constant function, so that $F'(x) = 0$.

If $g(x)$ is any differentiable but not constant function, is it true that we can always find a constant $\delta > 0$ such that $w(h)$ is well defined whenever h in $(-\delta, 0) \cup (0, \delta)$ (that is $0 < |h| < \delta$)?

The Proof of Chain Rule

Unfortunately, this guess is also INCORRECT!

There exists function which is differentiable at 0 but $w(h)$ is not well defined in $(-\delta, 0) \cup (0, \delta)$ for any $\delta > 0$.

Consider the function

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

In the case of $x = 0$, using squeeze theorem, we have

$$g'(x) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

Hence, g is differential at 0, which satisfies the condition of chain rule.

The Proof of Chain Rule

For $x = 0$, the function

$$w(h) = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}$$

is undefined means $g(h) - g(0) = 0$, that is $h^2 \sin \frac{1}{h} = 0$.

For $h = \frac{1}{n\pi}$ with any integer n, we have

$$h^2 \sin \frac{1}{h} = \frac{1}{(n\pi)^2} \sin(n\pi) = 0.$$

Hence, for any $\delta > 0$, we can take

$$n_0 = \left\lceil \frac{1}{\delta\pi} \right\rceil \quad \text{and} \quad h = \frac{1}{n_0\pi}$$

Then we have $h^2 \sin \frac{1}{h} = 0$ and $0 < h < \delta$.

The Proof of Chain Rule

The idea on Page 18 does NOT work!

$$
\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h\to 0} \left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \right] \\
&= \underbrace{\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}}_{\text{This term may be undefined!}} \cdot \underbrace{\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}}_{\text{slope of } g(x) \text{ at } x} \\
&= f'(g(x)) \cdot g'(x) \qquad \text{This step is WRONG!!!}
\end{aligned}
$$

SOS!!! How to Prove the Chain Rule???

$$\underbrace{\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}}_{\text{This term may be undefined!}} \cdot \underbrace{\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}}_{\text{slope of } g(x) \text{ at } x}$$

$$= f'(g(x)) \cdot g'(x)$$

This step is WRONG!!!

SOS!!! How to Prove the Chain Rule??????

We can replace the term

$$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}$$

by something which is well-defined!

The Proof of Chain Rule

We introduce a function as follows:

$$Q(y) = \begin{cases} \dfrac{f(y) - f(g(x))}{y - g(x)}, & y \neq g(x), \\ f'(g(x)), & y = g(x). \end{cases}$$

We can show

$$Q(g(x+h)) \cdot \frac{g(x+h) - g(x)}{h} \quad \text{is equal to} \quad \frac{f(g(x+h)) - f(g(x))}{h}.$$

1. Whenever $g(x+h)$ is not equal to $g(x)$, we have

$$\begin{aligned} & Q(g(x+h)) \cdot \frac{g(x+h) - g(x)}{h} \\ = & \frac{f(y) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} = \frac{f(g(x+h)) - f(g(x))}{h} \end{aligned}$$

2. When $g(x+h)$ equals $g(x)$, both of them are zero.

The Proof of Chain Rule

We have

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h\to 0} \left(Q(g(x+h)) \cdot \frac{g(x+h) - g(x)}{h} \right) \\ &= \lim_{h\to 0} Q(g(x+h)) \cdot \underbrace{\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}}_{g'(x)} \end{aligned}$$

Since both g and Q are continuous, we have

$$\lim_{h\to 0} Q(g(x+h)) = Q\left(\lim_{h\to 0} g(x+h)\right) = Q\left(g(x)\right) = f'(g(x)).$$

The Proof of Chain Rule

Let $u = g(x)$, we can verify the continuity of Q at $g(x)$:

$$\begin{aligned}\lim_{y\to g(x)} Q(y) = &\lim_{y\to u} Q(y)\\ =&\lim_{y\to u}\frac{f(y)-f(g(x))}{y-g(x)}\\ =&\lim_{y\to u}\frac{f(y)-f(u)}{y-u}\\ =&f'(u) = f'(g(x)) = Q(g(x)).\end{aligned}$$

Combing all above results, we can prove the chain rule

$$F'(x) = \lim_{h\to 0} Q(g(x+h)) \cdot \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(g(x))\cdot g'(x).$$

Examples of Chain Rule

Now, we can show $\dfrac{dx^p}{dx} = px^{p-1}$ for any constant exponent p.

Proof.

The definition of ln means $x^p = e^{\ln x^p} = e^{p \ln x}$, then

$$\frac{dx^p}{dx} = \frac{d\left(e^{p\ln x}\right)}{dx}$$

$$= \frac{de^u}{du} \cdot \frac{d\,(p\ln x)}{dx} \qquad \text{Chain rule with } f(u) = e^u \text{ and } g(x) = p\ln x$$

$$= e^u \cdot p \cdot \frac{d\ln x}{dx} \qquad \text{Using } \frac{de^u}{du} = e^u \text{ and } \frac{d\ln x}{x} = \frac{1}{x}$$

$$= x^p \cdot p \cdot \frac{1}{x} \qquad \text{Using } \frac{d\ln x}{x} = \frac{1}{x}$$

$$= px^{p-1}$$

□