Calculus IB: Lecture 08

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Outline

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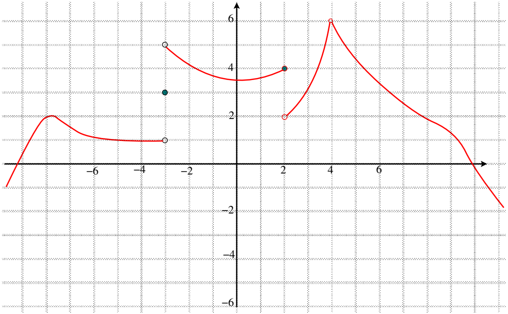
We have seen that even when $f(c)$, $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$ all exist, it is still possible that they are not equal.

When they are all well-defined and equal on real numbers (we do not consider ∞ or $-\infty$), we say that the function is *continuous* at $x = c$.

Roughly speaking, this is a mathematical way to say that no "sudden jump" on the graph will occur when passing through $x = c$.



Continuity of Functions

In this course, we focus on the continuity of functions defined on an interval, or the union of several intervals.

- If c is a real number in the domain of a function f such that a small open interval $(c-h, c+h)$ containing c, where $h > 0$, is entirely in the domain of f, c is called an *interior point* of the domain of f.

  A function $y = f(x)$ is said to be *continuous at an interior point* c in its domain if $\lim_{x \to c} f(x) = f(c)$.

- If a is a number in the domain of of f which is not an interior point, then the continuity condition $\lim_{x \to a} f(x) = f(a)$ should be understood as $f(x)$ is getting closer and closer to $f(a)$ as x in the domain of f is getting closer to closer to a.

  In particular, $x \to a$ should be understood as $x \to a^+$ is a is a "left endpoint" of the domain. Similarly, $x \to a$ should be understood as $x \to b^-$ if b is a "right endpoint" of the domain.

- Sometimes, d is called a *point of discontinuity* of a function f if the condition

$$\lim_{x \to a} f(x) = f(d)$$

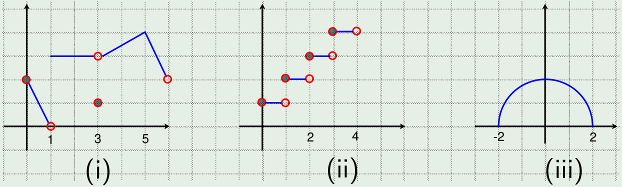
is not satisfied, i.e., either

  - $f(d)$ is not well-defined;
  - or or the limit does not exist at all;
  - or $f(d)$ is well-defined but not equal to the well-defined limit $\lim\limits_{x \to a} f(x)$.

According to this definition, every point not in the domain of f could be considered as a point of discontinuity of the function, which is sometime confusing.

Example

It is easy to see where the functions are continuous/discontinuous:



(i) Point of discontinuity: $x = 1, 3$, or 6. (Continuous at every point in the domain except $x = 1, 3$.)

(ii) Point of discontinuity: $x = 1, 2, 3$, or 4. (Continuous at every point in the domain except $x = 1, 2, 3$.)

(iii) Continuous at every point in the domain of the function.

(checking across which point the graph breaks into "separate pieces").

Continuity of Functions

Example

Find the value of the constant k such that the following piecewise polynomial function is continuous everywhere.

$$f(x) = \begin{cases} x^2 + 3x - 2k & \text{if } x \leq 1, \\ 2x - 3k & \text{if } x > 1. \end{cases}$$

It is easy to check that for any $a \neq 1$, $\lim_{x \to a} f(x) = f(a)$.

We now check the continuity condition of f at 1. Note that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 3x - 2k) = 4 - 2k = f(1)$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x - 3k) = 2 - 3k$$

To make f continuous at $x = 1$, we need to pick k so that $4 - 2k = 2 - 3k$, i.e., $k = -2$.

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- Sums, differences, products of continuous functions are continuous.
- In particular, polynomial functions are continuous on the entire real line. Recall here that a polynomial function of degree n is a function of the form

$$f(x) = ax^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers, and n is a non-negative integer.

- If two functions $f(x)$, $g(x)$ are continuous at $x = c$ and $g(c) \neq 0$, then the quotient $\frac{f}{g}$ is continuous at $x = c$.
- In particular, rational functions are continuous on the real line, except at the zeros of their denominators, i.e., continuous on their domains. Recall here that a *rational function* is a function of the form

$$f(x) = \frac{p(x)}{q(x)},$$

where $p(x), q(x)$ are polynomials with $q(x) \not\equiv 0$.
- For any positive integer n, the root function $f^{1/n}$ of a function f continuous at $x = c$ is also continuous at $x = c$, as long as the power function is well-defined on an open interval containing c.

Properties of Continuous Functions

These properties are straightforward consequences of the limit laws.

For example, if f and g are continuous at a, then

$$\lim_{x\to a} f(x) = f(a), \qquad \lim_{x\to a} g(x) = g(a)$$

and hence

$$\begin{aligned}
&\lim_{x\to a}(f+g)(x) \\
=&\lim_{x\to a}(f(x)+g(x)) \\
=&\lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\
=&f(a)+g(a) = (f+g)(a)
\end{aligned}$$

i.e., the function $f+g$ is also continuous at a.

The (ε, δ)-Definition of Continuity

Given a function f whose domain is D and an element x_0 in D, f is said to be continuous at the point x_0 when the following holds:

For any real number $\varepsilon > 0$, there exists some number $\delta > 0$ such that for all x in the domain of f with

$$|x - x_0| < \delta,$$

the value of $f(x)$ satisfies

$$|f(x) - f(x_0)| \leq \varepsilon.$$

The elementary functions $\sin x$, $\cos x$, $\tan x$, a^x and $\log_a x$ are all continuous at any point in their domains. We can check their graphs or prove the continuity by (ε, δ)-definition.

Properties of Continuous Functions

- Note also that if f is continuous at c and g is continuous at $f(c)$, then the composition of the two functions $g \circ f$ is continuous at c.
- In fact, as $x \to c$, $f(x) \to f(c)$ by the continuity of f at c, and hence $g(f(x)) \to g(f(c))$ by the continuity of g at $f(c)$.

Exercise

By drawing graphs, find some examples of f and g so that $g \circ f$ is continuous at c, while

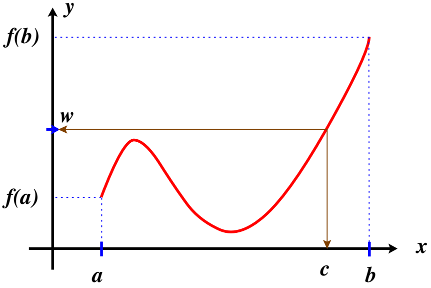
- g is not continuous at $f(c)$;
- or f is not continuous at c.

Outline

Intermediate Value Theorem

Theorem (Intermediate Value Theorem)

Suppose the function $y = f(x)$ is continuous on a closed interval $[a, b]$ and let w be a real number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there must be a number c in (a, b) such that $f(c) = w$.



In other words, the equation $f(x) = w$ must have at least one root in the interval (a, b). The Intermediate Value Theorem is very useful in locating roots of equations.

Example

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ in the interval $(1, 2)$.

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$, which is continuous on [1,2]. Then 0 is a number between $f(1)$ and $f(2)$:

$$-1 = f(1) < 0 < f(2) = 12.$$

By the Intermediate Value Theorem, there must be a number c in $(1, 2)$ such that $f(c) = 0$.

Similarly, $f(1.5) = 3.4 > 0$, hence the equation must have a root in the interval $(1, 1.5)$. We can also compute $f(1.25)$ to determine the root lies in $(1.1.25)$ or $(1.25, 1.5)$.

Continuing in this manner, one can end up with the "Bisection Method" for locating approximate roots of equations.

Intermediate Value Theorem

By the Intermediate Value Theorem, the problem of solving an inequality of the form $f(x) < 0$ for any continuous function f is essentially the same as solving $f(x) = 0$.

Once the zeros or undefined point of $f(x)$ are all located, it is just a matter of sign checking for $f(x)$ in various intervals in order to solve the inequality $f(x) < 0$ or $f(x) > 0$.

For example, the roots and undefined points of

$$\frac{(x+3)(2x-5)}{x+2} = 0$$

are $x = -3, \frac{5}{2}$ and -2 respectively, which divide the real line into four disjoint open intervals:

$$(-\infty, -3), \quad (-3, -2), \quad \left(-2, \frac{5}{2}\right), \quad \left(\frac{5}{2}, \infty\right) \; .$$

Note that $\frac{(x+3)(2x-5)}{x+2}$ cannot change sign in each of these intervals, since no other root is possible. By putting in some x values in these intervals, it is easy to see that

$$f(x) \begin{cases} < 0 & \text{if } x < -3 \text{ or } -2 < x < \dfrac{5}{2} \\ > 0 & \text{if } -3 < x < -2 \text{ or } x > \dfrac{5}{2} \end{cases}$$

Outline

Recall that the rate of change of a function $y = f(x)$ at $x = a$ is a certain limit called the *derivative of f at a*, which is denoted by $f'(a)$, and is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \overset{or}{=} \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

whenever the limit exists.

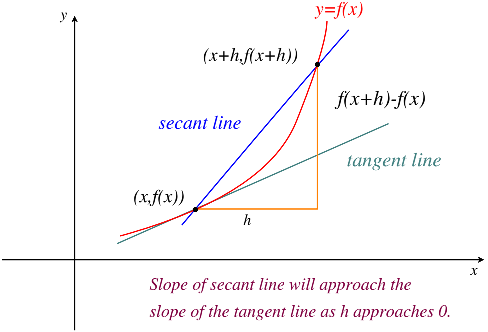
- The function f is said to be *differentiable at $x = a$* when $f'(a)$ exists on real numbers. (only correct for single variable calculus)
- Recall also that the limit $f'(a)$ can be interpreted as the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.

Limit Definition of Derivatives

If we want to measure how fast the function value $y = f(x)$ changes as x varies, we consider the *derivative function* $f'(x)$, which is defined as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

whenever the limit exists. Geometrically speaking, f' is the slope function of f.



Some other often used notations to denote the derivative $f'(x)$ of the function $y = f(x)$ are as follows:

$$\frac{df}{dx}, \quad \frac{dy}{dx}, \quad y', \quad \text{and} \quad \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a} = y'(a) = f'(a).$$

The process of finding the derivative of a given function is called differentiation.

When computing derivatives by using the limit definition of derivative, it is sometimes called differentiating by the first principle.

Example

Find the equation of the tangent line to the graph of the function $y = f(x) = 2x^2 - 3$ at the point $(1, -1)$.

The slope of the tangent line passing through $(1, -1)$ is

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[2(1+h)^2 - 3] - [2 \cdot 1^2 - 3]}{h}$$

$$= \lim_{h \to 0} \frac{2 + 4h + 2h^2 - 3 - 2 + 3}{h} = \lim_{h \to 0} (4 + 2h) = 4$$

Therefore the slope of the tangent line at $(1, -1)$ is 4, and the equation of the tangent line is given by

$$\frac{y - (-1)}{x - 1} = 4 \Longrightarrow y = 4x - 5$$

Examples of Derivatives

Example

Given function $y = f(x) = 2x^2 - 3$, find the derivative function $f'(x)$.

The derivative function $f'(x)$, by the limit definition of derivative (or the "first principle"), is given by the limit

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[2(x+h)^2 - 3] - [2x^2 - 3]}{h} \\ &= \lim_{h \to 0} \frac{4xh + 2h^2}{h} \\ &= \lim_{h \to 0} (4x + 2h) \\ &= 4x \end{aligned}$$

Examples of Derivatives

Example

Differentiate the function

$$f(x) = \frac{1}{x+2}$$

by using the limit definition of derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\dfrac{1}{(x+h)+2} - \dfrac{1}{x+2}}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{(x+2)-(x+h+2)}{(x+h+2)(x+2)}}{h} = \lim_{h\to 0} \frac{-1}{(x+h+2)(x+2)}$$

$$= -\frac{1}{(x+2)^2}$$

Examples of Derivatives

Example

Differentiate the function

$$g(x) = \sqrt{2x-1}$$

by using the limit definition of derivative.

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{2(x+h)-1}-\sqrt{2x-1}}{h} \\ &= \lim_{h\to 0}\frac{(\sqrt{2x+2h-1}-\sqrt{2x-1})(\sqrt{2x+2h-1}+\sqrt{2x-1})}{h(\sqrt{2x+2h-1}+\sqrt{2x-1})} \\ &= \lim_{h\to 0}\frac{2}{\sqrt{2x+2h-1}+\sqrt{2x-1}} \\ &= \frac{1}{\sqrt{2x-1}} \end{aligned}$$

Differentiable and Continuous

Theorem

If f is differentiable at a point $x = a$, then f is continuous at $x = a$.

Proof.

We have

$$\begin{aligned}
\lim_{x\to a} f(x) - f(a) &= \lim_{x\to a}(f(x) - f(a)) \\
&= \lim_{x\to a}\left[\frac{f(x) - f(a)}{x - a}\cdot(x - a)\right] \\
&= \lim_{x\to a}\frac{f(x) - f(a)}{x - a}\cdot\lim_{x\to a}(x - a) \\
&= f'(a)\cdot 0 = 0
\end{aligned}$$

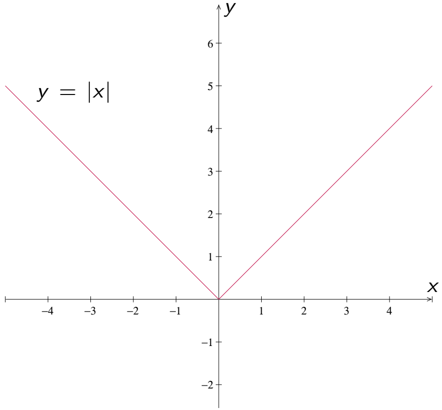
That is, $\lim_{x\to a} f(x) = f(a)$ and hence the function is continuous at a. □

Outline

Non-Differentiability

The derivative of a continuous function may not exist at every point.

A basic example is $f(x) = |x|$. Its derivative at $x = 0$, namely $f'(0)$, does not exist since there is no tangent line to the graph at $(0, 0)$.



Non-Differentiability

More precisely, by the limit definition of derivative, we have

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h| - 0}{h}$$

but

$$\lim_{h\to 0^+} \frac{|h| - 0}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1, \quad \lim_{h\to 0^-} \frac{|h| - 0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

i.e., the one-sided limits do not agree and therefore the limit does not exist.

Exercise

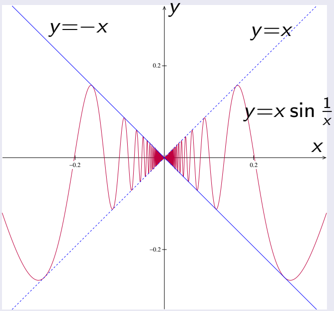
Show that $f(x) = |x|$ is differentiable at $x = x_0$ whenever $x_0 \neq 0$.

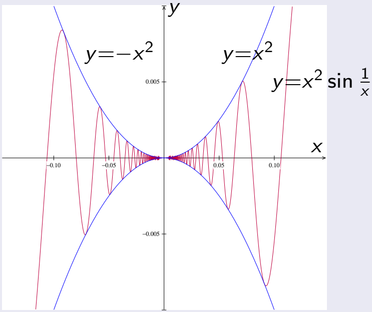
Non-Differentiability

Exercise

Show by working the limit definition of derivative that $f'(0)$ and $g'(0)$ do not exist where

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$





Weierstrass Function

The Weierstrass function is an example of a real-valued function that is continuous everywhere but differentiable nowhere.

In Weierstrass's original paper, the function was defined as follows:

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

where $0 < a < 1$, b is a positive odd integer and $ab > 1 + \frac{3}{2}\pi$.