Calculus IB: Lecture 07

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Outline

Outline

Examples of $\frac{\infty}{\infty}$ Type Limits

Example

Find the limit $\lim\limits_{x\to+\infty} \dfrac{2x^2 - x + 3}{3x^2 + x - 1}$.

We need to understand the behavior of the function $\frac{1}{x}$ as $x \to +\infty$:

$$\lim_{x\to+\infty} \frac{2x^2 - x + 3}{3x^2 + x - 1} = \lim_{x\to+\infty} \frac{x^2(2 - \frac{1}{x} + \frac{3}{x^2})}{x^2(3 + \frac{1}{x} - \frac{1}{x^2})}$$

$$= \lim_{x\to+\infty} \frac{2 - \frac{1}{x} + \frac{3}{x^2}}{3 + \frac{1}{x} - \frac{1}{x^2}} = \frac{2 - 0 + 3 \cdot 0}{3 + 0 - 0} = \frac{2}{3},$$

where we use the fact

$$\lim_{x\to+\infty} \frac{1}{x} = 0, \quad \lim_{x\to+\infty} \frac{1}{x^2} = \lim_{x\to+\infty} \frac{1}{x} \cdot \lim_{x\to+\infty} \frac{1}{x} = 0 \cdot 0 = 0.$$

Examples of $\frac{\infty}{\infty}$ Type Limits

Example

(a) $\lim\limits_{x\to+\infty} \dfrac{\sqrt{2x+1}-1}{x} = \lim\limits_{x\to+\infty} \left[\sqrt{\dfrac{2}{x}-\dfrac{1}{x^2}}-\dfrac{1}{x}\right] = 0.$

(b) $\lim\limits_{x\to+\infty} \dfrac{x^2}{\sqrt{x^2+4}-2} = \lim\limits_{x\to+\infty} \dfrac{x}{\sqrt{1+\frac{4}{x^2}}-\frac{2}{x}} = +\infty$

(c) $\lim\limits_{x\to+\infty} \dfrac{2x}{\sqrt{x^2+4}-2} = \lim\limits_{x\to+\infty} \dfrac{2x}{x(\sqrt{1+\frac{4}{x^2}}-\frac{2}{x})} = \lim\limits_{x\to+\infty} \dfrac{2}{\sqrt{1+\frac{4}{x^2}}-\frac{2}{x}} = 2$

Examples of $\infty - \infty$ Type Limits

Example

$$\begin{aligned}
&\lim_{x\to+\infty}(\sqrt{x+1}-\sqrt{x}) \\
=&\lim_{x\to+\infty}\frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{\sqrt{x+1}+\sqrt{x}} \\
=&\lim_{x\to+\infty}\frac{1}{\sqrt{x+1}+\sqrt{x}} \\
=&\ 0
\end{aligned}$$

$$\text{Why } \lim_{x\to+\infty}\frac{1}{\sqrt{x+1}+\sqrt{x}} = 0?$$

Examples of $\infty - \infty$ Type Limits

Example

$$\begin{aligned}
&\lim_{x\to+\infty}\left(\sqrt{x^2+x}-x\right)\\
=&\lim_{x\to+\infty}\frac{\left(\sqrt{x^2+x}-x\right)\left(\sqrt{x^2+x}+x\right)}{\left(\sqrt{x^2+x}+x\right)}\\
=&\lim_{x\to+\infty}\frac{x}{\left(\sqrt{x^2+x}+x\right)}\\
=&\lim_{x\to+\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}=\frac{1}{2}
\end{aligned}$$

Examples of $\infty - \infty$ Type Limits

When computing limits of the form

$$\lim_{x\to\infty}(f(x) - g(x)),$$

where both f and g are approaching ∞ as x is approaching ∞, one is actually looking at the trending behaviour of the gap between the graph of f and g, i.e., how

$$f(x) - g(x)$$

behaves as $x \to \infty$.

Examples of $\infty - \infty$ Type Limits

Example

Find one-sided limits: $\lim_{x\to 1^-} \frac{x^2 - x + 1}{x^2 - 1}$ and $\lim_{x\to 1^+} \frac{x^2 - x + 1}{x^2 - 1}$. We have

$$\lim_{x\to 1^-} \frac{x^2 - x + 1}{x^2 - 1} = \lim_{x\to 1^-} \frac{x^2 - x + 1}{x + 1} \cdot \frac{1}{x - 1} = \frac{1}{2} \cdot (-\infty) = -\infty$$

and

$$\lim_{x\to 1^+} \frac{x^2 - x + 1}{x^2 - 1} = \lim_{x\to 1^+} \frac{x^2 - x + 1}{x + 1} \cdot \frac{1}{x - 1} = \frac{1}{2} \cdot \infty = \infty$$

Hence $x = 1$ is a vertical asymptote of the function $\frac{x^2 - x + 1}{x^2 - 1}$.

Outline

Limit Laws with Infinity

We define the arithmetic operations as follows

$$c \cdot \infty = \infty \cdot c = \infty \quad \text{for real number } c > 0.$$

Based on above notations, we can generalize

$$\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

to the case that one of $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ is ∞.

Formally speaking, the notation $\infty \cdot c$ means we have

$$\lim_{x \to a} [f(x)g(x)] = \infty$$

when $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = c > 0$.

The Proof of $\infty \cdot c = \infty$

Proof: The limit $\lim_{x \to a} f(x) = \infty$ means for any $M_0 > 0$, there exists $\delta_0 > 0$ such that $f(x) > M_0$ whenever $0 < |x - a| < \delta_0$. Hence, for every $M > 0$, let $M_0 = \frac{2M}{c}$, there exists $\delta_0 > 0$ such that

$$|f(x)| = f(x) > \frac{2M}{c}$$

whenever $0 < |x - a| < \delta_0$.

The limit $\lim_{x \to a} g(x) = c$ means for every $\varepsilon_1 > 0$ there exists $\delta_1 > 0$ such that $|g(x) - c| < \varepsilon_1$ whenever $0 < |x - a| < \delta_1$. Let $\varepsilon_1 = \frac{c}{2}$, there exist $\delta_1 > 0$ such that we have

$$-\frac{c}{2} < g(x) - c < \frac{c}{2} \Longrightarrow \frac{c}{2} < g(x) < \frac{3c}{2} \Longrightarrow |g(x)| > \frac{c}{2}$$

whenever $0 < |x - a| < \delta_1$.

The Proof of $\infty \cdot c = \infty$

For every $M > 0$, there exists $\delta = \min(\delta_0, \delta_1)$ such that

$$|f(x)| > \frac{2M}{c} \text{ and } |g(x)| > \frac{c}{2} \Longrightarrow |f(x)g(x)| > M$$

whenever $0 < |x - a| < \delta$. The notation $\min(\delta_0, \delta_1)$ means the minimizer of δ_0 and δ_1.

Hence we can conclude

$$\lim_{x \to a}[f(x)g(x)] = \infty$$

when $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = c > 0$ by precise definition of limit.

Extended Real Number System

We introduce extended real number system to address the calculation contains the ∞ and $-\infty$. It is useful in describing the algebra on infinities and the various limiting behaviors in calculus.

Recall that $\mathbb{R} = (-\infty, \infty)$ presents the set of all real number.

The extended real number system is denoted by $\overline{\mathbb{R}}$ or $[-\infty, +\infty]$ or $\mathbb{R} \cup \{-\infty, +\infty\}$.

Here, "$+\infty$" is equivalent to "∞" and "$-(-\infty)$".

Arithmetic Operations on $\overline{\mathbb{R}}$

$$a + \infty = +\infty + a = +\infty, \qquad a \neq -\infty$$

$$a - \infty = -\infty + a = -\infty, \qquad a \neq +\infty$$

$$a \cdot (+\infty) = +\infty \cdot a = +\infty, \qquad a \in (0, +\infty]$$

$$a \cdot (-\infty) = -\infty \cdot a = -\infty, \qquad a \in (0, +\infty]$$

$$a \cdot (+\infty) = +\infty \cdot a = -\infty, \qquad a \in [-\infty, 0)$$

$$a \cdot (-\infty) = -\infty \cdot a = +\infty, \qquad a \in [-\infty, 0)$$

Arithmetic Operations on $\overline{\mathbb{R}}$

$$\frac{a}{+\infty} = \frac{a}{-\infty} = 0, \qquad a \in \mathbb{R}$$

$$\frac{+\infty}{a} = +\infty, \qquad a \in (0, +\infty)$$

$$\frac{-\infty}{a} = -\infty, \qquad a \in (0, +\infty)$$

$$\frac{+\infty}{a} = -\infty, \qquad a \in (-\infty, 0)$$

$$\frac{-\infty}{a} = +\infty, \qquad a \in (-\infty, 0)$$

Arithmetic Operations on $\overline{\mathbb{R}}$

$$a^{+\infty} = +\infty \qquad a \in (1, +\infty]$$

$$a^{-\infty} = 0 \qquad a \in (1, +\infty]$$

$$a^{+\infty} = 0 \qquad a \in [0, 1)$$

$$a^{-\infty} = +\infty \qquad a \in [0, 1)$$

$$0^{a} = 0 \qquad a \in (0, +\infty]$$

$$(+\infty)^{a} = +\infty \qquad a \in (0, +\infty]$$

$$(+\infty)^{a} = 0 \qquad a \in [-\infty, 0)$$

Correction: The rule $0^a = +\infty$ for $a \in [-\infty, 0)$ is **NOT** allowed in MATH 1013, just like $1/0$ we explain in next page.

Extended Real Number System

However, the following expressions are still undefined

$$\frac{+\infty}{+\infty} \qquad \frac{+\infty}{-\infty} \qquad \frac{-\infty}{+\infty} \qquad \frac{-\infty}{-\infty}$$

$$0\cdot(+\infty) \qquad 0\cdot(-\infty) \qquad (+\infty)\cdot 0 \qquad (-\infty)\cdot 0$$

$$\infty - \infty \qquad (-\infty)-(-\infty)$$

In the context of probability or measure theory, the product of 0 and ∞ (or $-\infty$) is often defined as 0, but it is **NOT** allowed in MATH 1013.

The expression $1/0$ (or $0^a = +\infty$ for $a \in [-\infty, 0)$) is still left undefined, since

$$\lim_{x\to 0^+}\frac{1}{x} = +\infty \neq -\infty = \lim_{x\to 0^-}\frac{1}{x}.$$

In contexts only non-negative values are considered, it is often convenient to define $1/0 = +\infty$. But it is **NOT** allowed in MATH 1013.

Extended Real Number System

We can extend following laws to extended real number system if all expressions are defined on $\overline{\mathbb{R}}$ based on above slides.

1. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$ for any constant c
2. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}[f(x)]^p = \left(\lim_{x\to a} f(x)\right)^p$ for any rational exponent p when $\left(\lim_{x\to a} f(x)\right)^p$ exists.

Exercise

Find (i) $\lim_{x \to -1^-} \frac{x^2 - x + 1}{x^2 - 1}$, (ii) $\lim_{x \to -1^+} \frac{x^2 - x + 1}{x^2 - 1}$. Can you find all vertical asymptotes of the function $\frac{x^2 - x + 1}{x^2 - 1}$? and any horizontal asymptotes?

Exercise

Compute the limit (a) $\lim_{x \to 0^+} \left(\frac{1}{\sqrt{x}} - \frac{1}{x} \right)$, (b) $\lim_{x \to 2} \sqrt{\frac{x^2 - 5x + 6}{x^2 - 4}}$.

Exercise

Compute the limit (a) $\lim_{x \to e^2} \frac{(\ln x)^3 - 8}{(\ln x)^2 - 4}$, (b) $\lim_{x \to 0} \frac{1 + \sin x}{\cos^2 x}$.

Outline

Squeeze Theorem

Squeeze Theorem (or Sandwich Theorem)

Let I be an interval having the point a. Let g, f, and h be functions defined on I, except possibly at a itself. Suppose that for every x in I NOT equal to a, we have If $g(x) \leq f(x) \leq h(x)$ for all x near a, except perhaps when $x = a$, then

$$\lim_{x \to a} g(x) \leq \lim_{x \to a} f(x) \leq \lim_{x \to a} h(x)$$

whenever these limits exist. (The same is true for one-sided limits.)

Exercise

Try to prove squeeze theorem by (ε, δ)-definition.

Note that we only require $g(x) \leq f(x) \leq h(x)$ holds locally.

Squeeze Theorem

Example

Suppose $1 - 2x^2 \leq f(x) \leq 1 + 3x^2$ for $-1 < x < 1$. Then by the Squeeze Theorem, we have

$$1 = \lim_{x \to 0}(1 - 2x^2) \leq \lim_{x \to 0} f(x) \leq \lim_{x \to 0}(1 + 3x^2) = 1$$

and hence

$$\lim_{x \to 0} f(x) = 1 \ .$$

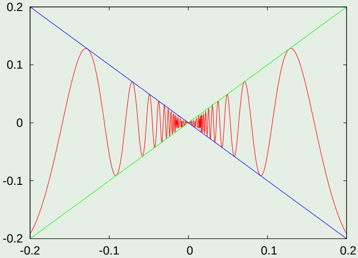
Squeeze Theorem

Example

Show that $\lim_{x\to 0} x \sin\frac{1}{x} = 0$ by applying the Squeeze Theorem:

$$-|x| \le x\sin\frac{1}{x} \le |x|$$

$$0 = -\lim_{x\to 0}|x| \le \lim_{x\to 0} x\sin\frac{1}{x} \le \lim_{x\to 0}|x| = 0\ .$$



Squeeze Theorem

Note that we CANNOT apply the limit law about product to write

$$\lim_{x\to 0} x \sin\frac{1}{x} = \lim_{x\to 0} x \cdot \lim_{x\to 0} \sin\frac{1}{x}$$

since $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist! (neither $+\infty$ nor $-\infty$)

Example

Show that $\lim_{t\to+\infty} e^{-t/2}\sin 5t = 0.$

Since $|\sin 5t| \leq 1$, we have

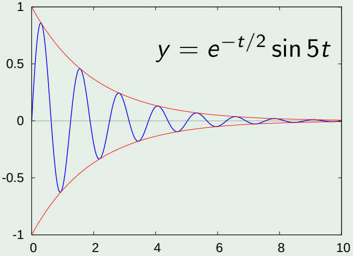
$$-e^{-t/2} \leq e^{-t/2}\sin 5t \leq e^{-t/2}.$$

On the other hand, we have

$$\lim_{t\to+\infty} e^{-t/2} = \lim_{t\to+\infty} -e^{-t/2} = 0.$$

Applying the Squeeze Theorem, then

$$\lim_{t\to+\infty} e^{-t/2}\sin 5t = 0.$$



Examples of Squeeze Theorem

In Lecture 04, we guessed

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$$

by calculating several points of θ near 0:

θ	0.1	0.01	0.001	0.0001
$\sin \theta / \theta$	0.998334166	0.999983333	0.999999833	0.999999998

Now, we can prove it by Squeeze Theorem.

Examples of Squeeze Theorem

By the Squeeze Theorem, this limit follows easily from the following inequalities: for $0 < \theta < \frac{\pi}{2}$,

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Hence

$$1 = \lim_{\theta\to 0^+} \cos\theta \;\le\; \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} \;\le\; \lim_{\theta\to 0^+} 1 \;=\; 1$$

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$$

Note that $\frac{\sin\theta}{\theta}$ is an even functions. Hence

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1 \;.$$

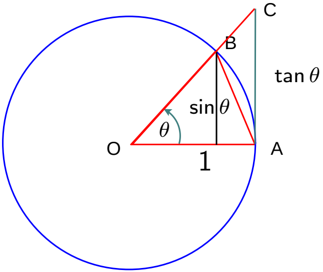
Examples of Squeeze Theorem

To prove $\cos\theta < \frac{\sin\theta}{\theta} < 1$ for $0 < \theta < \pi/2$, we compare the areas of following triangles and circular sector within unit circle:

$$\text{Area of } \triangle OAB < \text{Area of circular sector } OAB < \text{Area of } \triangle OAC$$

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta = \frac{\sin\theta}{2\cos\theta}$$

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$



Examples of Squeeze Theorem

Example

Using $\lim_{\theta\to 0}\frac{\sin k\theta}{k\theta}=1$ for any non-zero constant k, we have

(i) $\lim_{\theta\to 0}\frac{\tan 2\theta}{\theta}=\lim_{\theta\to 0}\left[\frac{\sin 2\theta}{2\theta}\cdot\frac{2}{\cos 2\theta}\right]=\lim_{\theta\to 0}\frac{\sin 2\theta}{2\theta}\cdot\lim_{\theta\to 0}\frac{2}{\cos 2\theta}=1\cdot 2=2$

(ii) $\lim_{x\to 0}\frac{\sin 3x}{2x}=\lim_{x\to 0}\frac{\sin 3x}{3x}\cdot\frac{3}{2}=\frac{3}{2}$

(iii) $\lim_{h\to 0}\frac{\cos h-1}{h}=\lim_{h\to 0}\frac{-2\sin^2\frac{h}{2}}{h}=-\lim_{h\to 0}\sin\frac{h}{2}\cdot\lim_{h\to 0}\frac{\sin\frac{h}{2}}{\frac{h}{2}}=0\cdot 1=0$

(we use $\cos h=1-2\sin^2(\frac{h}{2})$)

We can prove $\lim_{\theta\to 0}\frac{\sin k\theta}{k\theta}=\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ by (ε,δ)-definition.

Examples of Squeeze Theorem

Example

Given the inequality $e^x \geq x + 1$ for all x, show that $\lim_{x\to\infty} \frac{x}{e^x} = 0$.

Noting that $e^x = \left(e^{x/2}\right)^2 \geq \left(\frac{x}{2} + 1\right)^2 = \frac{x^2}{4} + x + 1$, we have for $x > 0$ the inequalities

$$0 < \frac{x}{e^x} \leq \frac{x}{\frac{x^2}{4} + x + 1}$$

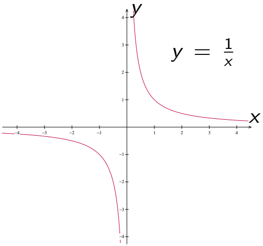
$$0 \leq \lim_{x\to\infty} \frac{x}{e^x} \leq \lim_{x\to\infty} \frac{x}{\frac{x^2}{4} + x + 1} = \lim_{x\to\infty} \frac{1}{\frac{x}{4} + 1 + \frac{1}{x}} = 0$$

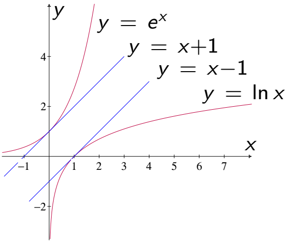
$$\lim_{x\to\infty} \frac{x}{e^x} = 0 .$$

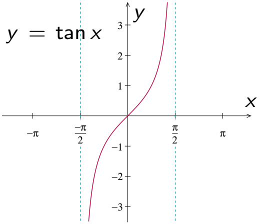
More generally, $\lim_{x\to\infty} \frac{x^n}{e^x} = 0$ for any positive integer n.

Why $e^x \geq x + 1$ holds for all x?

Summary of Some Basic Limits







$$\lim_{x\to\infty} \frac{1}{x} = 0 \qquad \lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to\frac{\pi}{2}^-} \tan x = \infty$$

$$\lim_{x\to-\infty} \frac{1}{x} = 0 \qquad \lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to-\frac{\pi}{2}^+} \tan x = -\infty$$

$$\lim_{x\to 0^+} \frac{1}{x} = \infty \qquad \lim_{x\to\infty} \ln x = \infty \qquad \lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$$

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty \qquad \lim_{x\to 0^+} \ln x = -\infty \qquad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Exercises of Squeeze Theorem

Exercise

Show that $\lim_{x \to 0^+} x \ln x = 0$ by letting $x = e^{-t}$.

Exercise

Show that $\lim_{x \to \infty} \dfrac{x^2}{e^x} = 0$, and hence $\lim_{t \to \infty} \dfrac{(\ln t)^2}{t} = 0$ by letting $x = \ln t$.