Calculus IB: Lecture 06

Luo Luo

Department of Mathematics, HKUST

http://luoluo.people.ust.hk/

Outline

Outline

Come back the requirement of MATH 1013!

An important point to keep in mind is that finding $\lim_{x \to a} f(x)$ is NOT the same as finding the function value $f(a)$.

1. $\lim_{x \to a} f(x)$ may exist even if $f(x)$ is undefined at $x = a$
2. $\lim_{x \to a} f(x)$ may not exists even if $f(x)$ is well-defined at $x = a$

Examples of Limits of Function Values

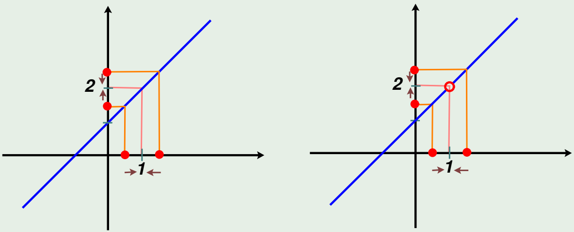
Example

Consider $f(x) = \dfrac{x^2 - 1}{x - 1}$ and $g(x) = x + 1$.

We have $\lim_{x \to 1} g(x) = \lim_{x \to 1} (x + 1) = 1 + 1 = 2 = g(1)$ and

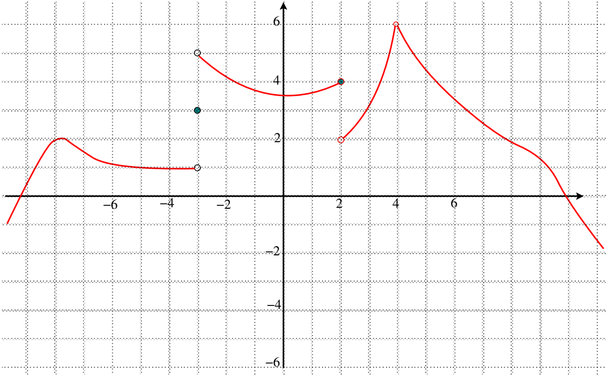
$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{(x-1)(x+1)}{x-1} = \lim_{x \to 1} (x+1) = 1 + 1 = 2,$$

but there is no well-defined function value $f(1)$.

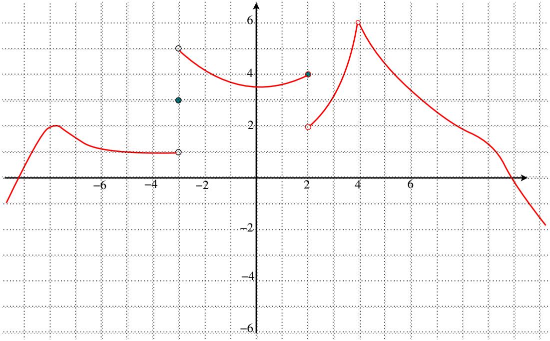


Finding Limits by Graphs

Graphically speaking, finding limits of function values is like riding along the graph (hollow circle means the function value is undefined at this point).



- $f(0)$ is well-defined, and $\lim_{x \to 0} f(x) = f(0)$.
- $\lim_{x \to 4} f(x) = 6$, while $f(4)$ is not well-defined.



- $f(-3) = 3$, but the left-hand limit is

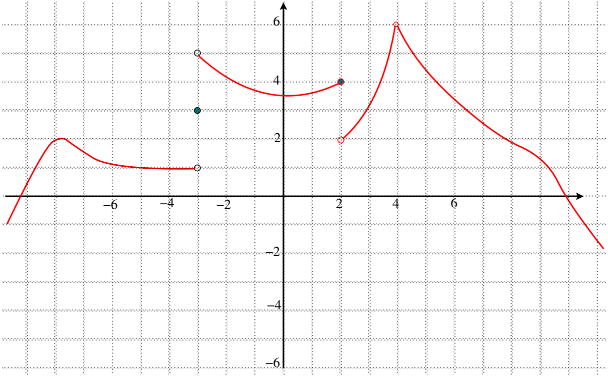
$$\lim_{x \to -3^-} f(x) = 1 \neq f(-3)$$

and the right-hand limit is

$$\lim_{x \to -3^+} f(x) = 5 \neq f(-3)$$

- $x \to -3^-$ means that x is approaching -3 from the left (i.e. $x < -3$)
- $x \to -3^+$ means that x is approaching -3 from the right (i.e. $x > -3$).

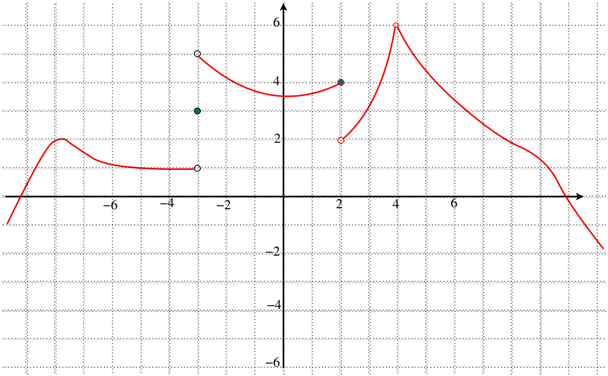
One-Side Limits



- Moreover, $\lim_{x \to -3} f(x)$ does not exist since

$$\lim_{x \to -3^-} f(x) = 1 \neq \lim_{x \to -3^+} f(x) = 5$$

One-Side Limits



- What happens as $x \to 2^-$, or $x \to 2^+$?
- We have $\lim_{x \to 2^-} f(x) = 4 = f(2)$, but $\lim_{x \to 2^+} f(x) = 2 \neq f(2) = 4$.
- The (two-sided) limit $\lim_{x \to 2} f(x)$ does not exist!

Limits and One-Side Limits

The limit $\lim_{x \to a} f(x)$ exists and equals the value L if and only if the two one-sided limits exist, and are equal to L:

$$\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

Example

Let $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$, then $f(0) = 0$, and

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (-1) = -1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$$

Since the two one-sided limits are not equal, $\lim_{x \to 0} f(x)$ does not exist.

Exercise

Sketch the graph of the following *piece-wise defined function*

$$f(x) = \begin{cases} x+2 & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ 2x+1 & \text{if } x > 3 \end{cases}$$

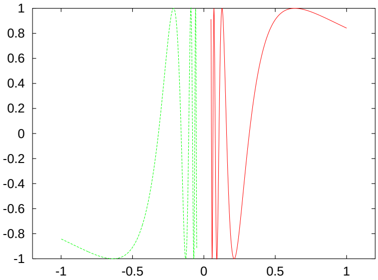
and find the one-sided limits $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$.

Does $\lim_{x\to 3} f(x)$ exist?

The function $f(x) = \sin\frac{\pi}{x}$ does not have any one-sided limit as $x \to 0^-$ or $x \to 0^+$.

The function value $f(x)$ keeps running up and down through the numbers between -1 and 1 without getting closer and closer to any fixed number when $x \to 0^-$, or $x \to 0^+$.

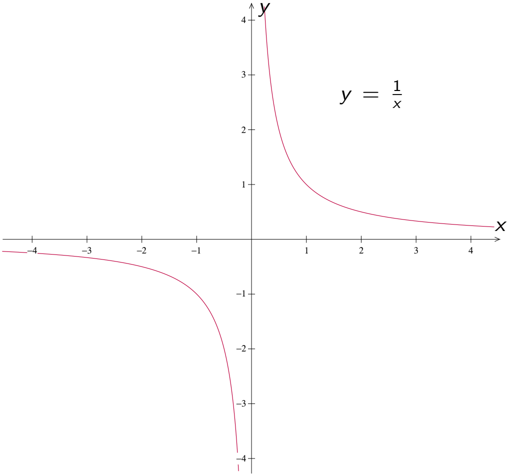
Note that $\sin\frac{\pi}{x} = 0$ whenever $\frac{\pi}{x} = n\pi$ for some integer n; i.e., whenever $x = \frac{1}{n}$ for some integer $n \neq 0$.



Outline

Limits of a Function $f(x)$ as $x \to \infty$ or $x \to -\infty$

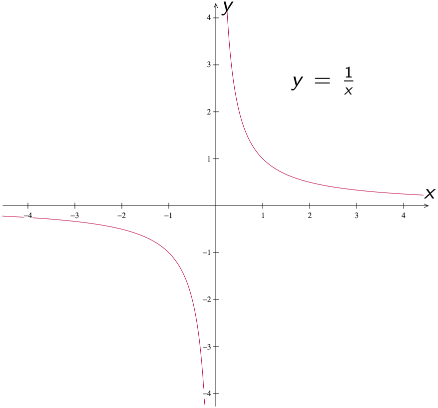
Consider the limit of the function $f(x) = \frac{1}{x}$ as $x \to 0^-, 0^+, -\infty, \infty$ or some constant $a \neq 0$. We can find these limits by graph of $f(x) = \frac{1}{x}$.



Limits of a Function $f(x)$ as $x \to \infty$ or $x \to -\infty$

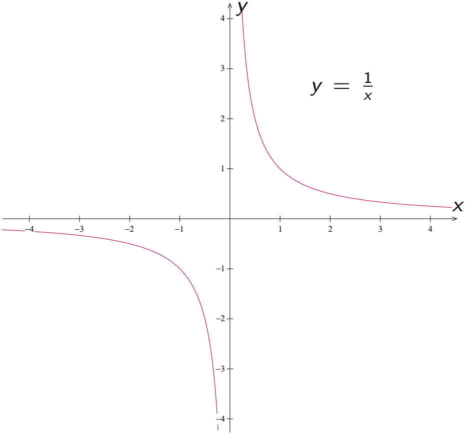
(a) $\displaystyle\lim_{x\to 0^+} \frac{1}{x} = +\infty$ (b) $\displaystyle\lim_{x\to 0^-} \frac{1}{x} = -\infty$ (c) $\displaystyle\lim_{x\to +\infty} \frac{1}{x} = 0$

(d) $\displaystyle\lim_{x\to -\infty} \frac{1}{x} = 0$ (e) $\displaystyle\lim_{x\to a} \frac{1}{x} = \frac{1}{a}$ for all real number $a \neq 0$



Horizontal Asymptote and Vertical Asymptote

The line $y = 0$ (x-axis) is called a *horizontal asymptote* of the function $f(x) = \frac{1}{x}$. The line $x = 0$ (y-axis) is called a *vertical asymptote* of this function.



Horizontal Asymptote and Vertical Asymptote

In general, we may consider the limiting behavior of $f(x)$ as $x \to \infty$ or $x \to -\infty$, or consider some one-sided limits to see if $f(x)$ is approaching ∞ or $-\infty$ as $x \to a^+$ or $a \to a^-$.

1. $y = L$ is a *horizontal asymptote* of the function $f(x)$ if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$.
2. $x = b$ is a *vertical asymptote* of the function $f(x)$ if at least one of the following holds:
    1. $\lim_{x\to b^-} f(x) = \infty$
    2. $\lim_{x\to b^-} f(x) = -\infty$
    3. $\lim_{x\to b^+} f(x) = \infty$
    4. $\lim_{x\to b^+} f(x) = -\infty$

Note that f has two different horizontal asymptotes $y = L_1$ and $y = L_2$ if

$$\lim_{x\to\infty} f(x) = L_1 \neq \lim_{x\to-\infty} f(x) = L_2$$

In any case, a function can have at most two horizontal asymptotes.

Horizontal Asymptote and Vertical Asymptote

Example

Find horizontal asymptote and vertical asymptote the function $f(x) = \dfrac{1}{x-2}$ by running along its graph.

(a) $\displaystyle\lim_{x\to+\infty} \frac{1}{x-2} = 0$

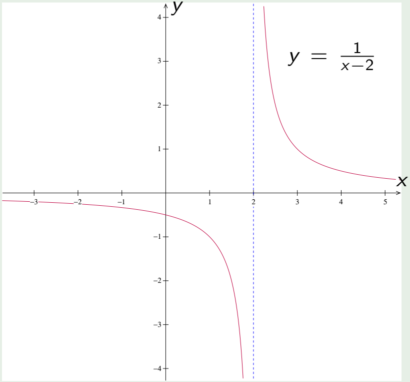
(b) $\displaystyle\lim_{x\to-\infty} \frac{1}{x-2} = 0$

(c) $\displaystyle\lim_{x\to 2^+} \frac{1}{x-2} = +\infty$

(d) $\displaystyle\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = 2$



Horizontal Asymptote and Vertical Asymptote

Example

Find horizontal asymptote and vertical asymptote the function $f(x) = \dfrac{x-1}{x} = 1 - \dfrac{1}{x}$ by running along its graph.

(a) $\displaystyle\lim_{x\to+\infty} \frac{x-1}{x} = 1$

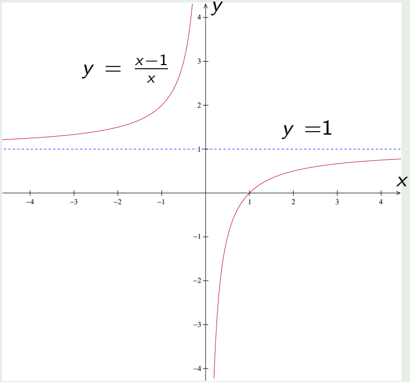
(b) $\displaystyle\lim_{x\to-\infty} \frac{x-1}{x} = 1$

(c) $\displaystyle\lim_{x\to 0^+} \frac{x-1}{x} = -\infty$

(d) $\displaystyle\lim_{x\to 0^-} \frac{x-1}{x} = +\infty$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 0$



Vertical Asymptote and Slant Asymptote

Summary of above results:

1. Given a function of the form $\dfrac{f(x)}{g(x)}$, the vertical line defined by $x = a$ is a vertical asymptote as long as $f(a) \neq 0$ (**Correction**: we do NOT require $f(x)$ is well-defined at a, but $f(a)$ cannot be 0 if it is well-defined) but $\lim_{x \to a^-} g(x) = 0$ or $\lim_{x \to a^+} g(x) = 0$.
2. If $f(x) = ax + b + g(x)$ with $g(x) \to 0$ as $x \to \infty$ or $x \to -\infty$, then the straightline given by $y = ax + b$ is called a *slant asymptote* of f.

Example of Slant Asymptote

Consider function $f(x) = \dfrac{x^2 + 2x + 3}{x} = x + 2 + \dfrac{3}{x}$.

(a) $x = 0$ is a vertical asymptote of f since

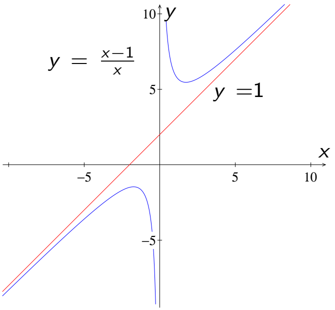
$$\lim_{x \to 0^+} \left(x + 2 + \frac{3}{x} \right) = \infty$$

$$\lim_{x \to 0^-} \left(x + 2 + \frac{3}{x} \right) = -\infty$$

(b) $y = x + 2$ is a slant asymptote of f since

$$\lim_{x \to \infty} (f(x) - (x + 2)) = \lim_{x \to \infty} \frac{3}{x} = 0$$

$$\lim_{x \to -\infty} (f(x) - (x + 2)) = \lim_{x \to -\infty} \frac{3}{x} = 0$$



Exercise of Slant Asymptote

Exercise

Show that $y = -x$ and $y = x$ are two slant asymptotes of the function $f(x) = \sqrt{1+x^2}$.

Hint: Consider the values of

$$\lim_{x\to\infty}\left(\sqrt{1+x^2}-x\right) = \lim_{x\to\infty}\left(\frac{(\sqrt{1+x^2}-x)(\sqrt{1+x^2}+x)}{\sqrt{1+x^2}+x}\right)$$

and

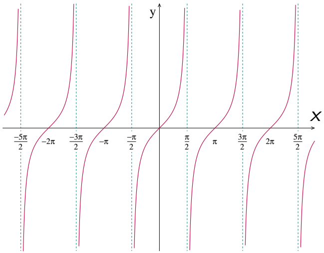
$$\lim_{x\to-\infty}\left(\sqrt{1+x^2}+x\right) = \lim_{x\to-\infty}\left(\frac{(\sqrt{1+x^2}+x)(\sqrt{1+x^2}-x)}{\sqrt{1+x^2}-x}\right).$$

Examples of Multiple Vertical Asymptotes

Consider the function $f(x) = \tan(x)$, we have

$$\lim_{x\to a^+} \tan(x) = -\infty \quad \text{and} \quad \lim_{x\to a^-} \tan(x) = \infty,$$

where $a = \frac{\pi}{2} + n\pi$ for any integer n.

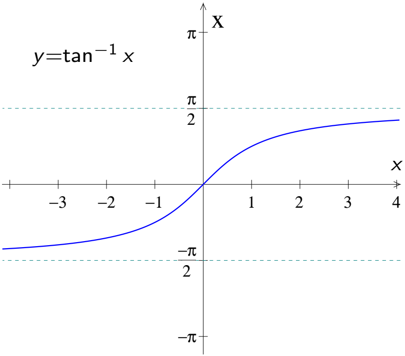


Hence $x = \frac{\pi}{2} + n\pi$ with any integer n is a vertical asymptote and there are infinite asymptotes in total.

Examples of Multiple Horizontal Asymptotes

Consider the function $f(x) = \tan^{-1}(x)$, we have

$$\lim_{x\to\infty} \tan(x) = \frac{\pi}{2} \quad \text{and} \quad \lim_{x\to-\infty} \tan(x) = -\frac{\pi}{2}.$$

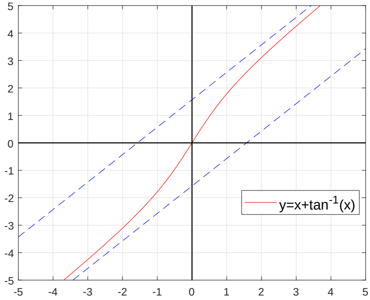


Hence $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are two horizontal asymptotes.

Examples of Multiple Slant Asymptotes

Consider the function $f(x) = x + \tan^{-1}(x)$, we have

$$\lim_{x\to\infty}\left(f(x) - x - \frac{\pi}{2}\right) = 0 \quad \text{and} \quad \lim_{x\to-\infty}\left(f(x) - x + \frac{\pi}{2}\right) = 0.$$



Hence $y = x - \frac{\pi}{2}$ and $y = x + \frac{\pi}{2}$ are two slant asymptotes.

Outline

Some Useful Limit Laws

Suppose that $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exists on *real numbers*, then we have:

1. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$ for any constant c
2. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
4. $\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$
6. $\lim_{x \to a} [f(x)]^p = \left(\lim_{x \to a} f(x) \right)^p$ for any rational exponent p when $(\lim_{x \to a} f(x))^p$ exists.

Some Useful Limit Laws

All of these rules can be proved by precise definition of limit, which is based on (ε, δ) language.

The following things are undefined:

$$\frac{\infty}{\infty}, \quad \frac{0}{0}, \quad 0 \cdot \infty \quad \text{and} \quad \infty - \infty.$$

Some Useful Limit Laws

Let $f(x) = \frac{1}{x^2}$ and $g(x) = -\frac{1}{x^2}$. What is $\lim_{x\to a}[f(x) + g(x)]$?

The definition of $f(x)$ and $g(x)$ means

$$f(x) + g(x) = \begin{cases} 0, & x \neq 0, \\ \text{undefined}, & x = 0. \end{cases}$$

Then we have $\lim_{x\to 0^+}[f(x) + g(x)] = \lim_{x\to 0^-}[f(x) + g(x)] = 0$ and

$$\lim_{x\to 0}[f(x) + g(x)] = 0.$$

However, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x\to 0} g(x) = \lim_{x\to 0} -\frac{1}{x^2} = -\infty.$$

Since $\infty + (-\infty)$ is undefined, we can NOT say $\infty + (-\infty) = 0$

Examples of Limits

After checking the existence of limit, we can use above rules.

Example

Find $\lim_{x\to 2}(x^2 - 2x + 5)$ and $\lim_{x\to 2} \sqrt[3]{x^2 - 2}$.

$$\begin{aligned}
&\lim_{x\to 2}(x^2 - 2x + 5) \\
=&\left(\lim_{x\to 2} x\right)^2 - 2\lim_{x\to 2} x + \lim_{x\to 2} 5 \\
=&2^2 - 2\cdot 2 + 5 = 7
\end{aligned}$$

$$\begin{aligned}
&\lim_{x\to 2} \sqrt[3]{x^2 - 2} \\
=&\sqrt[3]{\lim_{x\to 2}(x^2 - 2)} \\
=&\sqrt[3]{2^2 - 2} = \sqrt[3]{2}
\end{aligned}$$

Examples of Limits

Example

Find $\lim_{x\to 2} \dfrac{2x^2 - x + 1}{x^2 - 1}$.

$$\lim_{x\to 2} \frac{2x^2 - x + 1}{x^2 - 1}$$

$$= \frac{\lim_{x\to 2} \left(2x^2 - x + 1\right)}{\lim_{x\to 2}(x^2 - 1)}$$

$$= \frac{2 \cdot 2^2 - 2 + 1}{2^2 - 1} = \frac{7}{3}$$

Examples of Limits

Several algebraic tricks, mostly about factor canceling, are often needed in order to find limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example

Find the limit $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$. $\left(\dfrac{0}{0}\text{-type limit}\right)$

Note that if we directly substitute $x = 2$ into the expression, we will get some undefined expression $\frac{0}{0}$. This suggests that $(x - 2)$ is a factor of both the numerator and the denominator. After factoring, we have

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 2 + 2 = 4$$

Note that for $x \to 2$, we do not need to consider $x = 2$.

Examples of Limits

Example

Find the limit $\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9}$.

It is also a $\frac{0}{0}$ type limit. By suitable factor cancellation, we have

$$\begin{aligned}
&\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} \\
=&\lim_{x\to 9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} \\
=&\lim_{x\to 9} \frac{1}{\sqrt{x}+3} \\
=&\frac{1}{\sqrt{9}+3} = \frac{1}{6}.
\end{aligned}$$

More Examples of $\frac{0}{0}$ Type Limits

Here are some more examples of $\frac{0}{0}$ Type Limits, found by algebraic transformation.

Example

Find $\lim\limits_{x\to 0} \dfrac{\sqrt{2x+1}-1}{x}$.

$$\begin{aligned}
&\lim_{x\to 0} \frac{\sqrt{2x+1}-1}{x} \\
=&\lim_{x\to 0} \frac{\sqrt{2x+1}-1}{x} \cdot \frac{\sqrt{2x+1}+1}{\sqrt{2x+1}+1} \\
=&\lim_{x\to 0} \frac{2x}{x(\sqrt{2x+1}+1)} \\
=&\lim_{x\to 0} \frac{2}{\sqrt{2x+1}+1} = 1
\end{aligned}$$

More Examples of $\frac{0}{0}$ Type Limits

Example

Find $\lim_{x\to 0} \frac{x^2}{\sqrt{x^2+4}-2}$.

$$
\begin{aligned}
&\lim_{x\to 0} \frac{x^2}{\sqrt{x^2+4}-2} \\
=&\lim_{x\to 0} \frac{x^2}{\sqrt{x^2+4}-2} \cdot \frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2} \\
=&\lim_{x\to 0} \frac{x^2(\sqrt{x^2+4}+2)}{x^2} \\
=&4
\end{aligned}
$$