Calculus IB: Lecture 05

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Outline

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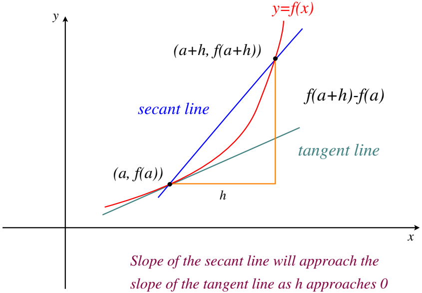
Limit Definition of Derivative

In general, given a function f, we can consider the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$ in a similar manner by looking at limiting behavior of the slopes of nearby secant lines:

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \triangleq f'(a) \text{ , whenever the limit exists.}$$

$f'(a)$ is called the *derivative of f at a*.



Examples of Derivative

Example

Let $f(x) = \dfrac{1}{x}$. Find the derivative $f'(2)$.

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{\frac{2-(2+h)}{(2+h)2}}{h} \\ &= \lim_{h\to 0} \frac{-1}{(2+h)2} = \frac{-1}{2\cdot 2} = -\frac{1}{4} \end{aligned}$$

where $f'(2) = -\frac{1}{4}$ can be interpreted as the slope of the tangent line to the graph of $y = \frac{1}{x}$ at the point $(2, \frac{1}{2})$.

Examples of Derivative

The term

$$\frac{f(a+h) - f(a)}{h}$$

is usually considered as the *average rate of change* of the function values of f over the interval $[a, a+h]$, and hence the limit $f'(a)$ is considered as the *instantaneous rate of change* of f at a.

Examples of Derivative

Example

Let $s(t) = t^2$ (in meters) be the position of a particle moving along the s-axis at time t (in seconds).

Consider that $t = 2.05$, then the average rate of change

$$\frac{(1+0.05)^2 - 1}{0.05} = 2.05 \ (\mathrm{m/s})$$

is the usual *average velocity* of the particle on the time interval $[1, 1.05]$.

The instantaneous rate of change of $s = t^2$ at $t = 1$ is called the *instantaneous velocity* of the particle at time $t = 1$:

$$s'(1) = \lim_{h\to 0} \frac{(1+h)^2 - 1}{h} = \lim_{h\to 0} \frac{h(2+h)}{h} = \lim_{h\to 0} 2 + h = 2 \ (\mathrm{m/s})$$

Outline

Limits of Function Values (Precise Definition)

This section is beyond the requirement of MATH 1013. It will NOT been contained in our homework or exam.

Intuitively speaking, given real numbers c and L, the expression

$$\lim_{x \to c} f(x) = L$$

means that $f(x)$ becomes arbitrarily close to L as x approaches c.

What is "arbitrarily close"? What is "approaches"?

The exact definition of $\lim_{x \to c} f(x)$ is still missing!

Limits of Function Values (Precise Definition)

The phrase "$f(x)$ becomes arbitrarily close to L" means that f(x) eventually lies in the interval $(L-\varepsilon, L+\varepsilon)$, which can also be written as $|f(x)-L|<\varepsilon$.

The phrase "as x approaches c" refer to values of x, whose distance from c is less than some positive number δ that is, values of x within either $(c-\delta, c)$ or $(c, c+\delta)$, which can be expressed with $0<|x-c|<\delta$.

The (ε,δ)-definition of limit: The expression $\lim_{x\to c} f(x) = L$ means for every $\varepsilon>0$ there exists $\delta>0$ such that $|f(x)-L|<\varepsilon$ whenever $0<|x-c|<\delta$ (it does NOT contain $x=c$).

1. It is possible that we cannot find any L such that $\lim_{x\to c} f(x) = L$.
2. We do not require $f(x)$ is well-defined at c, since $x=c$ does not satisfy $0<|x-c|<\delta$.

Example: Show that $\lim_{x \to 2} x^2 = 4$ by the (ε, δ)-definition.

Let $\varepsilon > 0$ be given; we want to show i.e., $|x^2 - 4| < \varepsilon$. How do we find δ such that any x satisfies $|x - 2| < \delta$, we are guaranteed that $|x^2 - 4| < \varepsilon$?

Note that $|x^2 - 4| = |x - 2| \cdot |x + 2|$, hence we consider (suppose $x \neq -2$)

$$|x^2 - 4| < \varepsilon \iff |x - 2| \cdot |x + 2| < \varepsilon \iff |x - 2| < \frac{\varepsilon}{|x + 2|}$$

We cannot set $\delta = \varepsilon / |x + 2|$, because given $\varepsilon > 0$, δ must be a constant value which is independent on x (δ can depends on ε).

Example: Show that $\lim_{x\to 2} x^2 = 4$ by the (ε, δ)-definition.

Recall that limit focus on local property of x^2 at $x = 2$, which implies that δ may be a small value. We can (probably) assume that $\delta < 1$. If this is true, then $|x-2| < \delta$ would imply that $|x-2| < 1$ and $1 < x < 3$, which implies $3 < x+2 < 5$. Then we have

$$\frac{1}{5} < \frac{1}{x+2} < \frac{1}{3} \Longrightarrow \frac{\varepsilon}{5} < \frac{\varepsilon}{x+2}$$

We want to keep $|x-2| < \dfrac{\varepsilon}{|x+2|}$. Then just take $\delta = \dfrac{\varepsilon}{5}$, we have

$$|x-2| < \delta \Longrightarrow |x-2| < \frac{\varepsilon}{5} < \frac{\varepsilon}{x+2} \Longrightarrow |x^2-4| < \varepsilon.$$

Recall that we have assume $\delta < 1$ (corresponds to $\varepsilon < 5$), which does not holds if $\varepsilon \geq 5$. Because in such case, we have $\delta = \varepsilon/5 \geq 1$.

Example: Show that $\lim_{x\to 2} x^2 = 4$ by the (ε, δ)-definition.

When $\varepsilon \geq 5$, we can just let $\delta = 1$. Similar to previous analysis for any $|x-2| < \delta = 1$, we have $-1 < x-2 < 1$ and $3 < x+2 < 5$, hence

$$|x^2 - 4| = |x-2| \cdot |x+2| < 1 \cdot 5 = 5 \leq \varepsilon \longrightarrow |x^2 - 4| < \varepsilon.$$

In summary, for every $\varepsilon > 0$, there exists $\delta = \min\left(\frac{\varepsilon}{5}, 1\right) > 0$ such that $|x^2 - 4| < \varepsilon$ whenever $0 < |x-2| < \delta$. Based on the (ε, δ)-definition of limit, we can conclude that

$$\lim_{x\to 2} x^2 = 4.$$

Limits of Function Values

We have provide the precise definition of

$$\lim_{x \to c} f(x) = L$$

when c and L are real numbers.

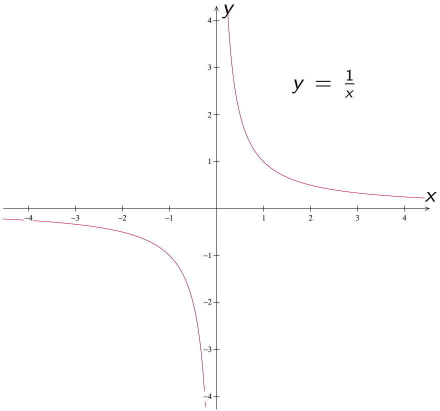
Recall the endpoints of intervals are not limited to real number. For example, we can define $(-\infty, 1]$, $(3, \infty)$, $(-\infty, \infty) \ldots$

What about infinity in limits?

Finite Limit at Infinity

Let $f(x) = \frac{1}{x}$. Then $f(x)$ becomes arbitrarily close to 0 as x is arbitrary large. We can write

$$\lim_{x \to \infty} f(x) = 0.$$



Finite Limit at Infinity

More general, given function $f(x)$ and real number L, the expression

$$\lim_{x \to \infty} f(x) = L$$

means for every $\varepsilon > 0$ there exists $N > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > N$.

Similarly, given function $f(x)$ and real number L, the expression

$$\lim_{x \to -\infty} f(x) = L$$

means for every $\varepsilon > 0$ there exists $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.

Finite Limit at Infinity

Example

Let $f(x) = \dfrac{1}{x}$. Show that $\lim_{x\to\infty} f(x) = 0$.

For every $\varepsilon > 0$, there exists $N = \dfrac{1}{\varepsilon}$ such that for any $x > N$, we have

$$|f(x) - 0| = \frac{1}{x} < \frac{1}{N} = \varepsilon.$$

Infinite Limit at Real Number

Given function $f(x)$ and real number c, the expression

$$\lim_{x \to c} f(x) = \infty$$

means for every $M > 0$ there exists $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$.

Similarly, given function $f(x)$ and real number c, the expression

$$\lim_{x \to c} f(x) = -\infty$$

means for every $M < 0$ there exists $\delta > 0$ such that $f(x) < M$ whenever $0 < |x - c| < \delta$.

Infinite Limit at Real Number

Exercise

Let $f(x) = \dfrac{1}{x^2}$. Using above definition to show that $\lim_{x \to 0} f(x) = \infty$.

Note that the result

$$\lim_{x \to c} f(x) = \infty \quad \text{or} \quad \lim_{x \to c} f(x) = -\infty,$$

implies the limit is undefined on real numbers.

However, the condition $\lim_{x \to c} f(x)$ is undefined on real numbers does NOT mean

$$\lim_{x \to c} f(x) = \infty \quad \text{or} \quad \lim_{x \to c} f(x) = -\infty,$$

Can you provide an example? (consider trigonometric functions)

Infinite Limit at Infinity

Given function $f(x)$, the expression

$$\lim_{x\to\infty} f(x) = \infty$$

means for every $M > 0$ there exists $N > 0$ such that $f(x) > M$ whenever $x > N$.

Similarly, we can also give the definitions of

1. $\lim_{x\to\infty} f(x) = -\infty$,
2. $\lim_{x\to-\infty} f(x) = \infty$,
3. $\lim_{x\to-\infty} f(x) = -\infty$.

The (ε, δ)-Definition

We can use (ε, δ)-definition to define a lot of things in calculus such as limit, continuity, derivative and integral.

This section is NOT contained in the requirement of MATH 1013. You can ignore this topic if you think it is difficult.

The (ε, δ)-definition is important to a deeper understanding of calculus. You can read a book of mathematical analysis if you are interested in it, but we will not talk about this topic in our course anymore.