Calculus IB: Lecture 04

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Outline

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More Trigonometric Functions

Four other trigonometric functions, namely, $\tan\theta$ (*tangent*), $\cot\theta$ (*cotangent*), $\csc\theta$ (*cosecant*), and $\sec\theta$ (*secant*) are defined by

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	domain: $\{\theta : \cos\theta \neq 0\}$ range: $(-\infty, \infty)$
$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	domain: $\{\theta : \sin\theta \neq 0\}$ range: $(-\infty, \infty)$
$\csc\theta = \dfrac{1}{\sin\theta}$	domain: $\{\theta : \sin\theta \neq 0\}$ range: $(-\infty, -1] \cup [1, \infty)$
$\sec\theta = \dfrac{1}{\cos\theta}$	domain: $\{\theta : \cos\theta \neq 0\}$ range: $(-\infty, -1] \cup [1, \infty)$

Properties of $\tan\theta$

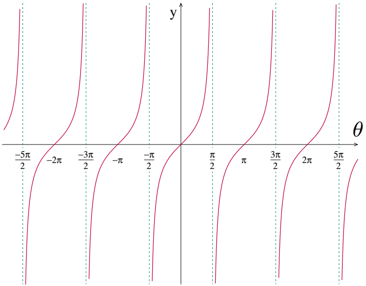
The function $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is a periodic function with period π:

$$\tan(\theta+\pi) = \tan\theta.$$

The domain of $\tan\theta$ is $\theta \neq n\pi + \pi/2$, where $n = 0, \pm 1, \pm 2, \pm 3, \cdots$, and the range is $(-\infty, \infty)$.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$--$

$\tan(-\theta) = -\tan\theta$ (odd function)



Properties of $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\csc\theta$

In addition to the identity $\sin^2\theta + \cos^2\theta = 1$, we have the identities:

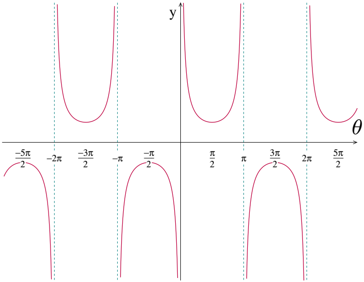
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

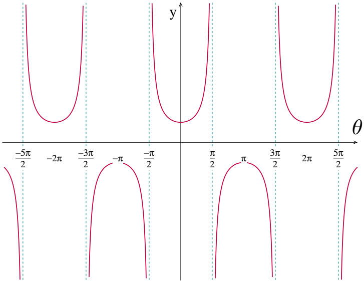
For example, we can prove

$$1 + \tan^2\theta = 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta.$$

You can prove the second one by yourself.

Graphs of $\csc\theta$ and $\sec\theta$





$$y = \csc\theta = \frac{1}{\sin\theta}$$

period $= 2\pi$

domain $\theta \neq n\pi$

where $n = 0, \pm1, \pm2, \cdots$

range $x \leq -1$ or $x \geq 1$

$$y = \sec\theta = \frac{1}{\cos\theta}$$

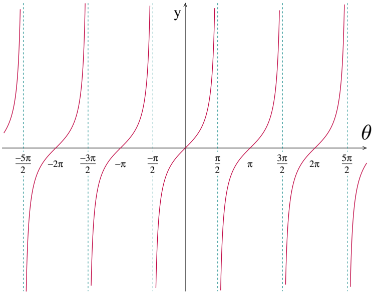
period $= 2\pi$

domain $\theta \neq n\pi + \frac{\pi}{2}$

where $n = 0, \pm1, \pm2, \cdots$

range $x \leq -1$ or $x \geq 1$

Graphs of csc θ and sec θ



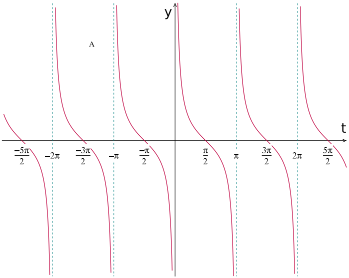
$$y = \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\text{period} = \pi$$

$$\text{domain } \theta \neq n\pi + \frac{\pi}{2}$$

$$\text{where } n = 0, \pm 1, \pm 2, \cdots$$

$$\text{range } (-\infty, +\infty)$$



$$y = \cot x = \frac{\cos\theta}{\sin\theta}$$

$$\text{period} = \pi$$

$$\text{domain } \theta \neq n\pi$$

$$\text{where } n = 0, \pm 1, \pm 2, \cdots$$

$$\text{range } (-\infty, +\infty)$$

Trigonometric Identities: Angle Addition and Subtraction

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

Trigonometric Identities: Product to Sum/Sum to Product

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha+\beta)-\cos(\alpha-\beta)\right]$$

$$\sin\alpha+\sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha-\sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha+\cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha-\cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Trigonometric Identities

All these formulas can be derived from one identity

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\beta\sin\alpha$$

For examples,

$$\begin{aligned} & \cos(\alpha - \beta) = \cos(\alpha + (-\beta)) \\ = & \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) \\ = & \cos\alpha\cos\beta + \sin\beta\sin\alpha \end{aligned}$$

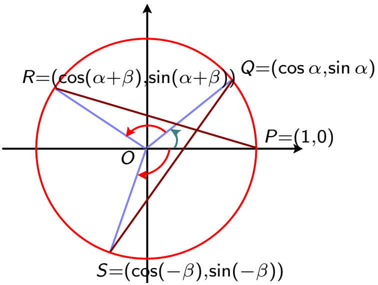
$$\begin{aligned} & \sin(\alpha - \beta) \\ = & \cos\left(\left(\frac{\pi}{2} - \alpha\right) + \beta\right) \\ = & \cos\left(\frac{\pi}{2} - \alpha\right)\cos\beta - \sin\left(\frac{\pi}{2} - \alpha\right)\sin\beta \\ = & \sin\alpha\cos\beta - \sin\beta\cos\alpha \end{aligned}$$

The Proof of $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\beta\sin\alpha$

Note that the two triangles $\triangle POR$ and $\triangle SOQ$ are congruent, as you can rotate one to the other by an angle of β. In particular, we have $PR = SQ$ and $PR^2 = SQ^2$.

Recall that the distance between two points (x_1, y_1) and (x_2, y_2) is given by

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



The identity follows then from $PR^2 = SQ^2$:

$$
\begin{aligned}
(\cos(\alpha+\beta)-1)^2 + (\sin(\alpha+\beta)-0)^2 &= (\cos\alpha - \cos(-\beta))^2 + (\sin\alpha - \sin(-\beta))^2 \\
\cos^2(\alpha+\beta) - 2\cos(\alpha+\beta) + 1 + \sin^2(\alpha+\beta) &= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta \\
&\quad + \sin^2\alpha + 2\sin\alpha\sin\beta + \sin^2\beta \\
2 - 2\cos(\alpha+\beta) &= 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta
\end{aligned}
$$

Some Exercises

1. Work out the triple angle formulas for $\sin 3\alpha$, $\cos 3\alpha$.
   Hint: $\sin 3\alpha = \sin(\alpha + 2\alpha)$
2. Can you rewrite functions like $y = a\sin\omega t + b\cos\omega t$ into the form $y = R\sin(\omega t + C)$ for some constants R, ω, C? For example, since

   $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, we have

   $$y = \sin t + \cos t = \sqrt{2}(\sin t\cos\frac{\pi}{4} + \cos t\sin\frac{\pi}{4}) = \sqrt{2}\sin(t + \frac{\pi}{4}) \ .$$

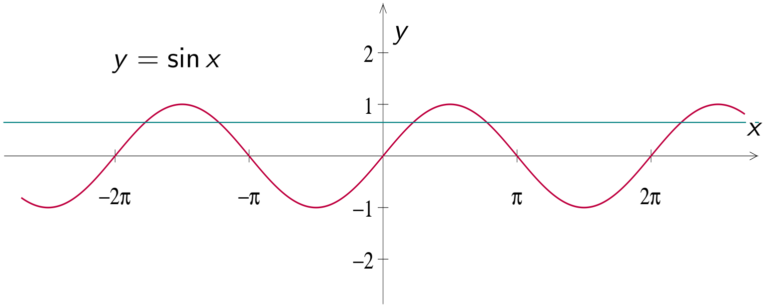
   Hint: consider

   $$a\sin\omega t + b\cos\omega t = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin\omega t + \frac{b}{\sqrt{a^2+b^2}}\cos\omega t\right]$$

   which can be rewritten as $R\sin(\omega t + C)$, or $R\cos(\omega t + C)$ for suitable choice of C.

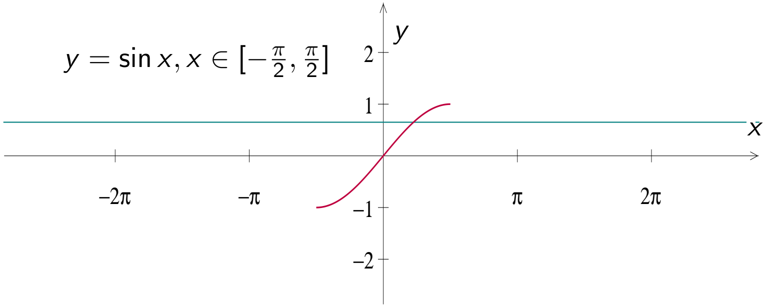
Outline

The horizontal line test shows that $\sin x$, $\cos x$, or $\tan x$ have no inverse function in general. For example, the periodic function $\sin x$ is obviously not one-to-one.



Inverse Trigonometric Functions

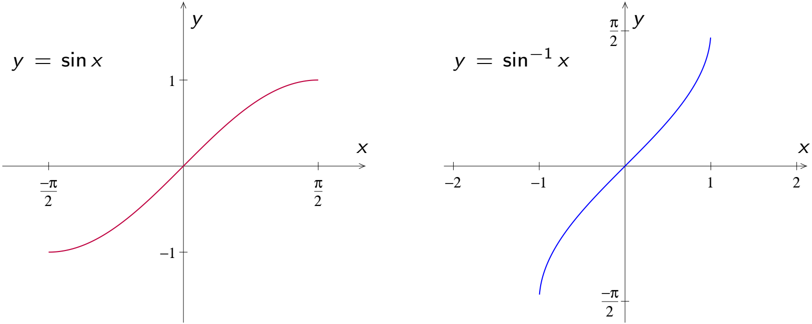
However, after restricting the domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the function $y = \sin x$ is one-to-one, and hence has an inverse function.



The inverse function of $y = \sin x$, with x restricted to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, is denoted by $y = \sin^{-1} x$ or $y = \arcsin x$.

Graphs of $\sin\theta$ and $\sin^{-1}\theta$

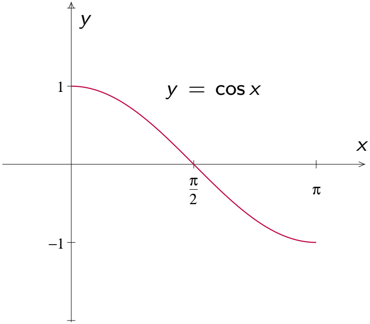
Recall that the graph of $y = \sin^{-1} x$ can be found by reflecting the part of the graph of $y = \sin x$, with $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, across the line $y = x$.

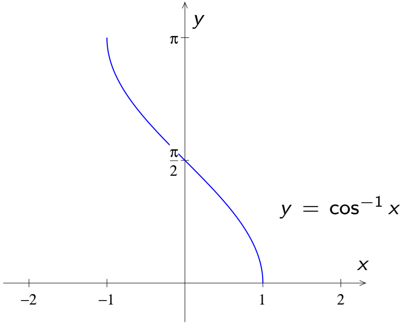


We can shift the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in theoretical, but we always use $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in definition by convention.

Graphs of $\cos\theta$ and $\cos^{-1}\theta$

The inverse trigonometric functions $\cos^{-1} x$ can also be defined by inverting the functions $\cos x$ with domain restricted to $0 \le x \le \pi$.



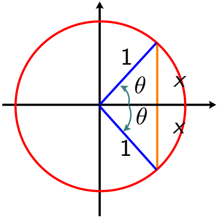


Inverse Trigonometric Functions

Another way to look at these inverse trigonometric functions is to consider solutions of trigonometric equations:

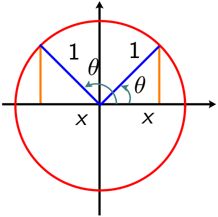
- $\sin^{-1} x$ is the unique solution θ (angle in radian measure) of the equation $x = \sin\theta$ chosen within the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (solvable for any $-1 \leq x \leq 1$).
- $\cos^{-1} x$ is the unique solution θ (angle in radian measure) of the equation $x = \cos\theta$ chosen within the closed interval $[0, \pi]$ (solvable for any $-1 \leq x \leq 1$).
- $\tan^{-1} \theta$ is the unique solution θ (angle in radian measure) of the equation $x = \tan\theta$ chosen within the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (solvable for any $-\infty < x < \infty$).

Graphical View of Solving Trigonometric Equations



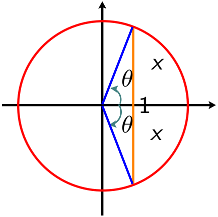
$-\frac{\pi}{2} \leq \theta = \sin^{-1} x \leq \frac{\pi}{2}$

$-1 \leq x \leq 1$



$0 \leq \theta = \cos^{-1} x \leq \frac{\pi}{2}$

$-1 \leq x \leq 1$



$-\frac{\pi}{2} \leq \theta = \tan^{-1} x \leq \frac{\pi}{2}$

$-\infty < x < \infty$

General Solution of Trigonometric Equations

Using the inverse trigonometric functions, one can express the general solutions of some basic trigonometric equations as follows:

$$\sin x = a \quad \begin{cases} x = n\pi + (-1)^n \sin^{-1} a & \text{if} \quad -1 < a < 1 \\ x = 2n\pi + \frac{\pi}{2} & \text{if} \quad a = 1 \\ x = 2n\pi - \frac{\pi}{2} & \text{if} \quad a = -1 \\ \text{no solution} & \text{if} \quad |a| > 1 \end{cases}$$

$$\cos x = a \quad \begin{cases} x = 2n\pi \pm \cos^{-1} a & \text{if} \quad -1 \leq a \leq 1 \\ \text{no solution} & \text{if} \quad |a| > 1 \end{cases}$$

$$\tan x = a \qquad x = n\pi + \tan^{-1} a \qquad \text{for any real number } a$$

where $n = 0, \pm 1, \pm 2, \pm 3, \cdots$ goes through the set of all integers.

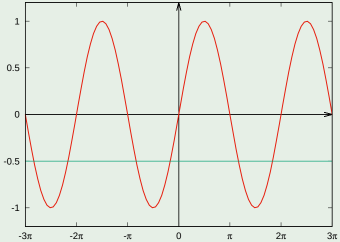
These formulas are based on the fact that the general solutions of trigonometric equations can be found from one known particular solution and periodic properties of trigonometric functions.

Examples of Solving Trigonometric Equations

Example

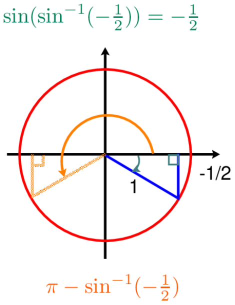
Find the general solution of the equation $\sin x = -\frac{1}{2}$.

We have $x = n\pi + (-1)^n \sin^{-1}\left(-\frac{1}{2}\right) = n\pi - (-1)^n \frac{\pi}{6}$, where $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ is a special solution of the equation, $n = 0, \pm 1, \pm 2, \ldots$



Examples of Solving Trigonometric Equations

We can also consider the unit circle:



Adding integer multiples of the periods to generate all solutions:

$$x = \begin{cases} 2n\pi + \sin^{-1}\left(-\frac{1}{2}\right) \\ 2n\pi + \pi - \sin^{-1}\left(-\frac{1}{2}\right) \\ n = 0, \pm 1, \pm 2, \ldots \end{cases} \iff \begin{cases} x = n\pi + (-1)^n \sin^{-1}\left(-\frac{1}{2}\right) \\ n = 0, \pm 1, \pm 2, \ldots \end{cases}$$

Outline

The Slope of a Tangent Line

In geometry, the *tangent line* to a curve at a given point is the straight line that "just touches" the curve at that point.

The *secant line* of a curve is a line that intersects the curve at a minimum of two distinct points.

Recall that the slope of a straight line passing through two distinct points (x_1, y_1), (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Let's consider the function $y = f(x) = x^2$. How to find the slope $m_{\tan}$ of the tangent line to the graph of f at the point $(1, 1)$?

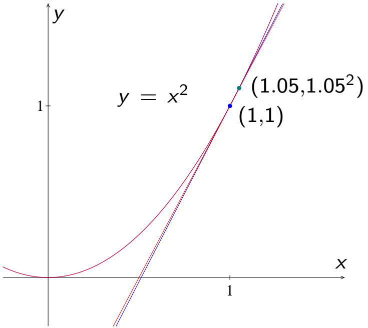
The Slope of a Tangent Line

How to find the slope $m_{\tan}$ of the tangent line to the graph of $y = f(x) = x^2$ at the point $A = (1, 1)$?

In addition to the point $A = (1, 1)$, we can take a nearby point $B = (1.05, 1.05^2)$ on the graph of $y = x^2$.

We can find the slope $m_{\sec}$ of the secant line of the graph which passes through these two points:

$$m_{sec} = \frac{1.05^2 - 1}{1.05 - 1} = 2.05 \approx m_{\tan}.$$



The Slope of a Tangent Line

We can get better and better approximation of the slope m_{tan} by looking at slope of secant line through $(1, 1)$ and another point $(1 + h, (1 + h)^2)$ on the graph, when h is chosen to be closer and closer to 0:

h	0.05	0.005	0.0005	0.00005
m_{sec}	2.05	2.005	2.0005	2.00005

In general, for any $h \neq 0$, we have

$$m_{\text{sec}} = \frac{(1+h)^2 - 1}{(1+h) - 1} = \frac{2h + h^2}{h} = 2 + h$$

Note that as $h \to 0$ ("as $h \neq 0$ is approaching 0"), m_{sec} is approaching the number 2, which gives us the slope of the tangent line $m_{\text{tan}} = 2$.

The Slope with Limit Notation

In terms of the "limit notation", this process can be written as

$$\lim_{h \to 0} \frac{(1+h)^2 - 1}{(1+h) - 1} = 2 = m_{\tan}$$

The equation of the tangent line to the graph of $y = x^2$ at the point $(1, 1)$ is then give by

$$\frac{y-1}{x-1} = 2 \Longleftrightarrow y = 2x - 1$$

In fact, it is easy to check that the straight line given by $y = 2x - 1$ intersects the graph of $y = x^2$ at exactly the point $(1, 1)$ by solving the equation

$$x^2 = 2x - 1 \Longleftrightarrow x^2 - 2x + 1 = (x-1)^2 = 0 \Longleftrightarrow x = 1 \ .$$

Example (tangent line problem of cubic function)

Find the equation of the tangent line to the curve defined by the equation $y = x^3$ at the point $(2, 8)$.

Consider the slope of the secant line passing through the point $(2, 8)$ and a nearby point $(2 + h, (2 + h)^3)$ on the curve. Then

$$m_{\text{sec}} = \frac{(2+h)^3 - 8}{(2+h) - 2} = \frac{12h + 6h^2 + h^3}{h} = 12 + 6h + h^2 \longrightarrow 12 \quad \text{as } h \to 0$$

Using the limit notation, we have

$$\lim_{h\to 0} \frac{(2+h)^3 - 8}{(2+h) - 2} = \lim_{h\to 0} \frac{12h + 6h^2 + h^3}{h} = \lim_{h\to 0}(12 + 6h + h^2) = 12$$

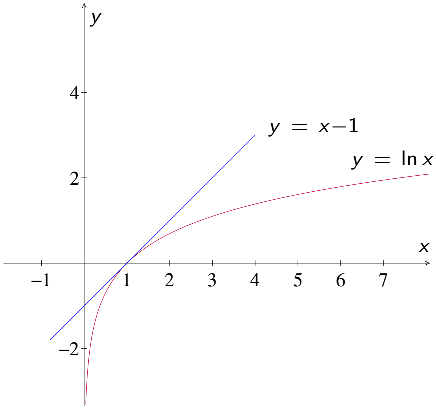
Hence the slope of the tangent line is 12, and the equation of the tangent line to the graph of $y = x^3$ at $(2, 8)$ is

$$\frac{y-8}{x-2} = 12 \iff y = 12x - 16$$

Outline

Why $y = e^x, y = \log_e x \triangleq \ln x$ with $e \approx 2.7182$?

One condition that determines the number e, which is the *base of the natural logarithmic function*, is that the slope of the tangent line to the graph of the natural logarithmic function $y = \log_e x = \ln x$ at $(1, 0)$ is 1.



Suppose that the slope of the tangent line to the graph of $y = \log_e x$ at the point $(1, 0)$ is 1. We regard e as an unknown number we want to find.

Then the trending behavior of the slope of the secant line passing through the point $(1, 0)$ and a nearby point $(1 + h, \log_e(1 + h))$ on the graph as $h \to 0$ should be

$$\begin{aligned} m_{\text{sec}} =& \frac{\log_e(1+h) - 0}{(1+h) - 1} = \frac{1}{h}\log_e(1+h) \\ =& \log_e(1+h)^{\frac{1}{h}} \longrightarrow 1 \quad \text{as } h \to 0 \end{aligned}$$

Using the limit notation, e is the number which satisfies

$$\lim_{h \to 0} \log_e(1+h)^{\frac{1}{h}} = 1.$$

Since we have $\log_e e = 1$, one way to define the number e is

$$e = \lim_{h \to 0} (1+h)^{\frac{1}{h}}.$$

Let $h = 10^{-10}$, then $(1 + 10^{-10})^{10^{10}} = 2.7182821\cdots \approx e = 2.7182818\cdots$

Tangent Line Problem of Exponential Function

Find the slope of the tangent line to the graph of the natural exponential function $y = e^x$ at the point $(0, 1)$.

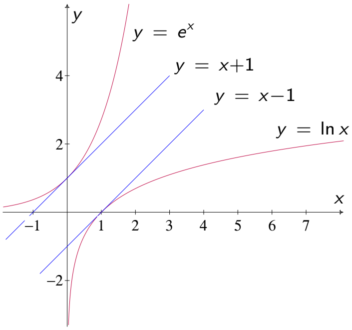
Just recall that the graph of $y = e^x$ can be found by reflecting the graph of its inverse function $y = \ln x$ across the line $y = x$.

The tangent line to the graph of $y = \ln x$ at the point $(1, 0)$ will be reflected to the tangent line to the graph of $y = e^x$ at the point $(0, 1)$.

It is easy to see that the slope of this tangent line to $y = e^x$ is also 1.

Using two nearby points $(0, 1)$, (h, e^h) on the graph of $y = e^x$, and the slope of the secant line passing through them, we have

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$



Tangent Line Problem of Exponential Function

What about the slope of the tangent line to the graph of the natural exponential function $y = e^x$ at the point (a, e^a)?

We have shown that the slope is 1 when $a = 0$. How to extend the result?

Note that the slope of the tangent line to the graph of $y = e^x$ at the point (a, e^a) can then be found by the trending behavior of the slope of the secant line through (a, e^a) and $(a + h, e^{a+h})$

$$\lim_{h \to 0} \frac{e^{a+h} - e^a}{h} = \lim_{h \to 0} \frac{e^a(e^h - 1)}{h} = e^a \cdot \lim_{h \to 0} \frac{e^h - 1}{h} = e^a \cdot 1 = e^a \, .$$

Tangent Line Problem of Sine Function

Find the slope of the tangent line to the graph of the function $y = \sin x$ at the point $(0, 0)$.

By considering the slope of the secant line through the points $(0, 0)$ and $(h, \sin h)$, the slope of the tangent line to the graph of the function $y = \sin x$ at the point $(0, 0)$ is given by

$$\lim_{h \to 0} \frac{\sin h - \sin 0}{h} = \lim_{h \to 0} \frac{\sin h}{h}$$

By calculating a few function values of $\frac{\sin h}{h}$ as $h \to 0$, it is reasonable to *guess* that $\lim_{h \to 0} \frac{\sin h}{h} = 1$.

x	0.1	0.01	0.001	0.0001
$\sin h/h$	0.998334166	0.999983333	0.999999833	0.999999998

The slope of the tangent line to the graph of $y = \sin x$ at the origin $(0, 0)$ is then equal to 1. The equation of the tangent line is $y = x$.

Tangent Line Problem of Sine Function

A precise explanation for

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

would require more understanding on the limits of function values. We shall look at this limit again later.