Calculus IB: Lecture 03

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Outline

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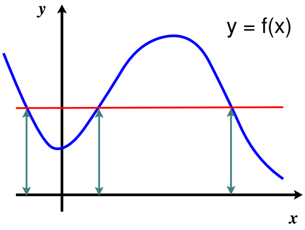
One-to-One Functions

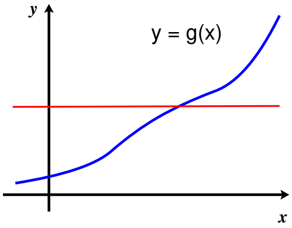
1. Recall that for any x in the domain of a function f, only one function value $f(x)$ can be assigned to x.
2. However, it is possible that two different numbers $x_1 \neq x_2$ in the domain of f have the same function value, i.e. $f(x_1) = f(x_2)$.
3. By ruling out above possibility, we have the concept of a *one-to-one function*: A function f is said to be *one-to-one* if $f(x_1) \neq f(x_2)$ for *any* two numbers $x_1 \neq x_2$ in the domain of f.
4. In other words, $f(x)$ never takes on the same function value twice or more times when x runs through the domain of f; or equivalently, the equation
$$f(x) = b$$
has exactly one solution for any b in the range of f. In particular, f is a one-to-one function if $x_1 = x_2$ whenever $f(x_1) = f(x_2)$.

One-to-One Functions

Graphically speaking, we have the *Horizontal Line Test* which says that f is a one-to-one function if every horizontal line hits the graph of f at most once.



f is not one-to-one : several x-values can produce the same y-value



g is one-to-one : different x-values can not produce the same y-value

Examples of One-to-One Functions

Example

Let f be defined by $f(x) = x^2$. It is obviously that f takes on every positive number $b \neq 0$ exactly two times, since $x^2 = b$ has exactly two roots $x = \pm\sqrt{b}$ for any $b > 0$; e.g.,

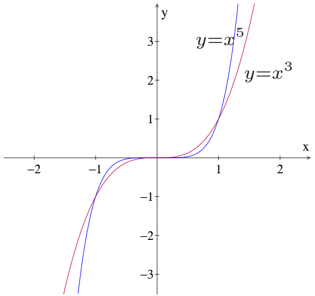
$$f(2) = 2^2 = 4 = (-2)^2 = f(-2) \; .$$

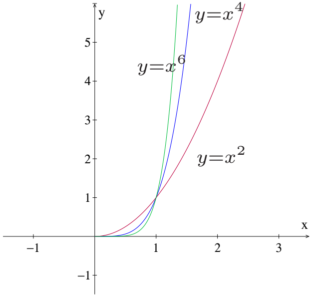
f is thus not a one-to-one function.

However, if the domain of $f(x) = x^2$ is *restricted* to $0 \leq x < \infty$, the function f is then one-to-one, since $x^2 = b$ has exactly one non-negative solution $\sqrt{b}$.

Increasing (Decreasing) Functions are One-to-One

1. If f is an increasing function with domain D a set of real numbers, then $f(x_1) < f(x_2)$ for any number x_1, x_2 in the domain D such that $x_1 < x_2$. Hence $f(x_1) \neq f(x_2)$ for any $x_1 \neq x_2$.
2. Similarly, the decreasing functions are also one-to-one functions.
3. For any positive integer n, the *power function* $y = x^{2n+1}$ is an increasing function with domain $-\infty < x < \infty$. Similarly, the power function $y = x^{2n}$ is an increasing function when the domain is restricted to $0 \leq x < \infty$.





Outline

Power Functions

Note that for any positive integer n, the function $\dfrac{1}{x^n}$ can also be expressed in the form of power function as $\dfrac{1}{x^n} = x^{-n}$.

The *exponent laws* for integer powers (or exponents) then follow easily:

$$\text{(i)}\;\; x^0 = 1 \text{ (by convention)} \qquad \text{(ii)}\;\; x^{n+m} = x^n x^m \qquad \text{(iii)}\;\; x^{n-m} = \frac{x^n}{x^m}$$

$$\text{(iv)}\;\; (x^n)^m = x^{nm} \qquad \text{(v)}\;\; (xy)^n = x^n y^n \qquad \text{(vi)}\;\; \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

where n, m are any integers.

Power Functions

For example, if n, m are positive integers with $n < m$, then

$$x^n x^m = (\underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ many factors}}) \cdot (\underbrace{x \cdot x \cdot \cdots \cdot x}_{m \text{ many factors}}) = \underbrace{x \cdot x \cdot \cdots \cdot x}_{n+m \text{ many factors}} = x^{n+m}$$

$$\frac{x^n}{x^m} = \frac{\overbrace{x \cdot x \cdot \cdots \cdot x}^{n \text{ many factors}}}{\underbrace{x \cdot x \cdot \cdots \cdot x}_{m \text{ many factors}}} = \frac{1}{\underbrace{x \cdot x \cdot \cdots \cdot x}_{m-n \text{ many factors}}} = \frac{1}{x^{m-n}} = x^{n-m}$$

Note that these exponent laws hold also for exponents which are real numbers. However, it would be harder to see what x^p means when p is a irrational number (such as $p = \sqrt{2}$).

Outline

Inverse Functions Arising from One-to-One Functions

Consider the linear relation

$$y = 2x + 3$$

between x and y, where y is considered as a function of x, can be rewritten as

$$x = \frac{y - 3}{2}$$

Hence x can then be considered as a function of y.

The same process can be applied to any one-to-one functions.

Inverse Functions Arising from One-to-One Functions

If f is a one-to-one function, then for any b in the range of f, the equation $f(x) = b$ has exactly one solution in the domain of f.

We can therefore define *inverse function* of f, usually denoted by f^{-1} (Warning: the symbol f^{-1} here does not mean $\frac{1}{f}$), by reversing the roles of the domain and range of f as follows:

$$f^{-1} : \begin{array}{c} \text{range of } f \\ \| \\ \text{domain of } f^{-1} \end{array} \longrightarrow \begin{array}{c} \text{domain of } f \\ \| \\ \text{range of } f^{-1} \end{array}$$

where

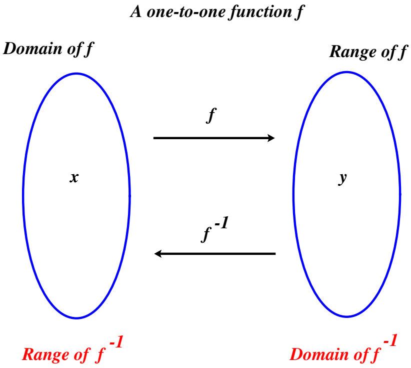
$$f^{-1}(b) = \text{the unique solution of the equation } f(x) = b$$

for any b in the domain of f^{-1} (i.e., the range of f).

Inverse Functions and Arrow Diagram

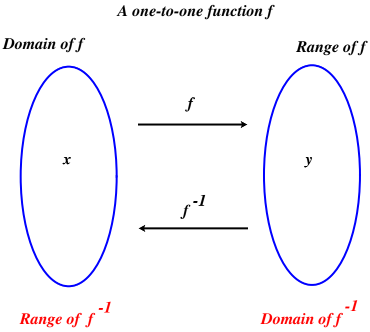
Suppose we use an arrow diagram to represent a function f, which assigns to any given number x in the domain of f a unique number y in the range of f (i.e., $y = f(x)$). Then, defining f^{-1} is just like "reversing the arrow" of f:

- turning the range of f into the domain of the inverse function f^{-1};
- turning the domain of f into the range of the inverse function f^{-1};
- $x = f^{-1}(y)$ coming from the unique solution of $f(x) = y$



Inverse Functions and Arrow Diagram

1. A one-to-one function $y = f(x)$ gives rise to a one-to-one matching of the numbers in two sets.
2. Depending on which variable you take as independent variable, you have either the original function $f(x)$, or the inverse function $f^{-1}(y)$.
3. The following properties of f and f^{-1} follow easily from chasing the arrows: $f^{-1}(f(x)) = x$ for any x in the domain of f; and $f(f^{-1}(y)) = y$ for any y in the range of f.



Examples of Inverse Functions

Example

Find the inverse function $f^{-1}(x)$ for the function $f(x) = \dfrac{3x+2}{2x-1}$. Let $y = \dfrac{3x+2}{2x-1}$, then $y(2x-1) = 3x+2 \Longleftrightarrow (2y-3)x = y+2$.

Hence we have

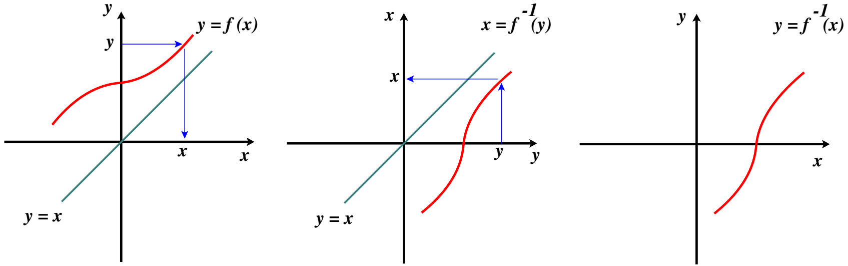
$$x = \frac{y+2}{2y-3} = f^{-1}(y) \quad \textit{and} \quad f^{-1}(x) = \frac{x+2}{2x-3}.$$

The domain of f^{-1}, which is the range of f, is given by $x \neq \dfrac{3}{2}$; i.e., $\left(-\infty, \dfrac{3}{2}\right) \bigcup \left(\dfrac{3}{2}, \infty\right)$. And the range of f^{-1}, which is the domain of f, is given by $x \neq \dfrac{1}{2}$; i.e., $\left(-\infty, \dfrac{1}{2}\right) \bigcup \left(\dfrac{1}{2}, \infty\right)$

Graphs of Inverse Functions

It is interesting that the graph of $x = f^{-1}(y)$ is the same as the graph of $y = f(x)$, except that the y-axis is now viewed as the domain axis.

In particular, the graph of the inverse function $y = f^{-1}(x)$ can be obtained by reflecting the graph of the one-to-one function $y = f(x)$ across the line $y = x$, or simply by renaming the x-axis as the y-axis, and y-axis as the x-axis.



Reflecting range into domain

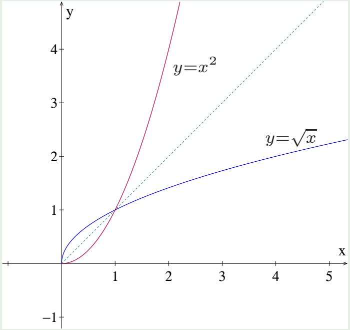
Renaming the axes

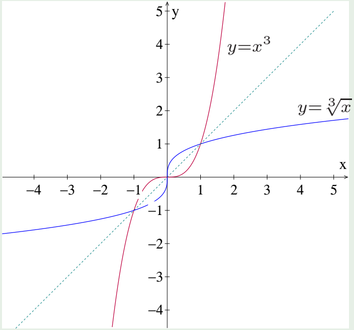
Graphs of Inverse Functions

Example

For any integer $n \geq 2$, the *n-th root function* $\sqrt[n]{x}$ is defined as follows:

$$\sqrt[n]{x} = \begin{cases} \text{the inverse function of } y = x^n \text{ with domain } [0, \infty) & \text{if } n \text{ is even} \\ \text{the inverse function of } y = x^n & \text{if } n \text{ is odd} \end{cases}$$





In particular, the domain of an n-th root function is given as follows.

$$\text{domain of } \sqrt[n]{x} \text{ is given by: } \begin{cases} [0, \infty) & \text{if } n \text{ is even} \\ (-\infty, \infty) & \text{if } n \text{ is odd} \end{cases}$$

Using exponent notation, an n-th root function can be written as

$$\sqrt[n]{x} = x^{\frac{1}{n}}.$$

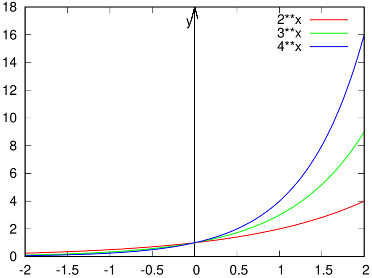
More generally, a *power function* of the form $x^{\frac{n}{m}}$, where n is an integer and m is a positive integer, is defined by

$$x^{\frac{n}{m}} = \sqrt[m]{x^n}\ .$$

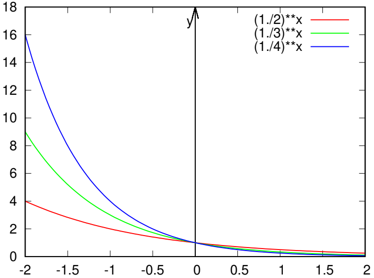
Outline

Exponential and Logarithmic Functions

For any positive real number $a \neq 1$, the *exponential function with base a* is given by $y = a^x$, whose graphs are shown as follows.



$y = 2^x,\ y = 3^x,\ y = 4^x$
$(a > 1)$



$y = (\frac{1}{2})^x,\ y = (\frac{1}{3})^x,\ y = (\frac{1}{4})^x$
$(0 < a < 1)$

1. The domain of $y = a^x$ is $(-\infty, \infty)$.
2. The range of $y = a^x$ is $(0, \infty)$.
3. We also have

$$y = a^x = \begin{cases} \text{is an increasing function} & \text{if } a > 1, \\ \text{is a decreasing function} & \text{if } 0 < a < 1. \end{cases}$$

4. Since many expressions with negative a like $(-1)^{1/2}$ it not a real number, and since $a = 0$ leads to a trivial constant function, we usually only consider the case of $a > 0$.

Exponential and Logarithmic Functions

An exponential function $y = a^x$ must be one-to-one (try to prove it), and hence has an inverse function, which is denoted by $x = \log_a y$, by reversing the roles of the domain and range:

$$\begin{cases} y = a^x \\ \text{domain: } -\infty < x < \infty \\ \text{range: } y > 0 \end{cases}$$

$$\longleftrightarrow \begin{cases} x = \log_a y \\ \text{domain: } y > 0 \\ \text{range: } -\infty < x < \infty \end{cases}$$

$$\longleftrightarrow \begin{cases} y = \log_a x \\ \text{domain: } x > 0 \\ \text{range: } -\infty < y < \infty \end{cases}$$

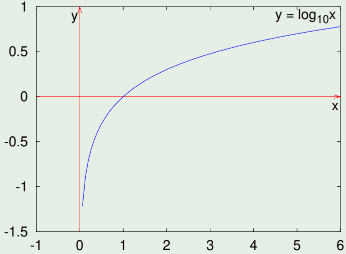
Example (take $a = 10$ and see what happens)

$y = 10^x \downarrow$

x	-3	-2	-1	0	$c = ?$	1	2	$b = ?$
y	0.001	0.01	0.1	1	8	10	100	100

$\uparrow x = \log_{10}(y)$

The graph of the *exponential function with base 10*, $y = 10^x$, gives you the graphs of the *common logarithmic function* $y = \log_{10} x$ at the same time.



Note that to find the value of $b = \log_{10} 100$ is just a problem of solving the equation $10^b = 100 = 10^2$, and hence obviously $b = 2 = \log_{10} 100$.
It is not so easy to find the exact value of $c = \log_{10} 8$ though, which means $10^c = 8$. A rough estimate is $\frac{1}{2} < c < 1$ since $10^{1/2} < 8 = 10^c < 10^1$.

Properties of Exponential and Logarithmic Functions

Once you understand how to convert exponential relationship into logarithmic relationship, and vice versa,

$$y = a^x \quad \longleftrightarrow \quad x = \log_a y$$

the following properties of logarithms are easy to verify.

Exponential Function	Logarithmic Function
$a^0 = 1$	$\log_a 1 = 0$
$a^1 = a$	$\log_a a = 1$
$a^x = a^x$	$\log_a a^x = x$
$a^{\log_a x} = x$	$\log_a x = \log_a x$
$a^x a^y = a^{x+y}$	$\log_a xy = \log_a x + \log_a y$
$\frac{a^x}{a^y} = a^{x-y}$	$\log_a \frac{x}{y} = \log_a x - \log_a y$
$(a^x)^y = a^{xy}$	$\log_a x^y = y \log_a x$
	$\log_c x = \frac{\log_a x}{\log_a c}$

Verify the property $\log_a(xy) = \log_a x + \log_a y$

Let $C = \log_a x$, and $D = \log_a y$. Hence we have $a^C = x$ and $a^D = y$. What if you multiplying the two together?

$$a^C a^D = xy \iff a^{C+D} = xy$$

Now, convert it to:

$$\log_a(xy) = C + D = \log_a x + \log_a y$$

All other properties in the table above can be checked by similar arguments. (Exercise!)

The Natural Exponential/Logarithmic Function

The exponential/logarithmic function with a special base $e \approx 2.7182\ldots$

$$y = e^x, \qquad y = \log_e x \triangleq \ln x$$

is called the *natural exponential/logarithmic function*. Note that all other exponential function can be expressed in term of the natural exponential function, since

$$a^x = e^{\ln a^x} = e^{x \ln a} \ .$$

For example, $3^x = e^{x \ln 3}$.

The Natural Exponential/Logarithmic Function

Why are we interested in the very special number $e \approx 2.7182\ldots$?

More precisely, e can be defined as the "limit" as follows

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

We shall discuss the topic about "e" in more detail later.

Example

Find the domain and range of the function $y = f(x) = 2\ln(5-x)+1$. What is its inverse function?

Recall that $\log_a(\bigstar)$ is well-defined if and only if $\bigstar > 0$. Hence the domain of $f(x)$ is given by: $5-x>0$, i.e., $x<5$. We also have

$$\begin{aligned} & y = 2\ln(5-x)+1 \\ \Longrightarrow & \frac{y-1}{2} = \ln(5-x) \\ \Longrightarrow & 5-x = e^{\frac{y-1}{2}} \\ \Longrightarrow & x = 5-e^{\frac{y-1}{2}} \end{aligned}$$

i.e., the inverse function $x = f^{-1}(y) = 5 - e^{\frac{y-1}{2}}$. The range of $f(x)$ is the domain of $f^{-1}(y)$, which is the set of all real numbers.

More Examples on Using Exp-Log

Example

Solve the following equations: (a) $24(1 - e^{-t/2}) = 16$; (b) $2^{2x-3} = 3^{x+1}$

$$24(1 - e^{-t/2}) = 16$$
$$1 - e^{-t/2} = \frac{16}{24} = \frac{2}{3}$$
$$e^{-t/2} = \frac{1}{3}$$
$$-\frac{t}{2} = \ln\frac{1}{3}$$
$$t = -2\ln\frac{1}{3} = \ln 9 \quad (\approx 2.1792)$$

$$2^{2x-3} = 3^{x+1}$$
$$\ln 2^{2x-3} = \ln 3^{x+1}$$
$$(2x-3)\ln 2 = (x+1)\ln 3$$
$$(2\ln 2 - \ln 3)x = \ln 3 + 3\ln 2$$
$$x = \frac{\ln 3 + 3\ln 2}{2\ln 2 - \ln 3}$$

Hyperbolic Functions

The hyperbolic functions are defined and denoted by

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

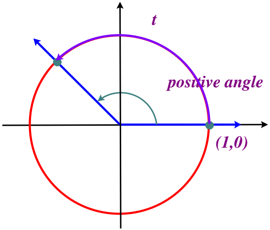
Please verify the following identities:

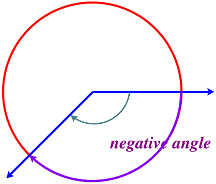
- $\cosh^2 x - \sinh^2 x = 1$
- $\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$
- $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$

Hyperbolic functions have some similar properties to *trigonometric functions*.

Outline

If the point $(1, 0)$ starts to travel along the unit circle centered at the $(0, 0)$ through a distance θ in counterclockwise direction, the angle subtended by the corresponding circular arc is said to be a positive angle with *radian measure* θ.
Angles obtained by clockwise rotations are considered as negative angles.





Directed angle : angle can be assigned a +ve or -ve sign

Radian Measure of an Angle

Radian is a measure of an angle by circular arc length along the unit circle.

1. Recall that the length of a unit circle is 2π. Thus the radian measure of a 360° angle is 2π, and -2π if the angle is -360°.
2. In proportion, the degree measure and radian measure of an angle can be converted to each other according to

$$\frac{\text{radian measure}}{\text{degree measure}} = \frac{2\pi}{360} = \frac{\pi}{180}$$

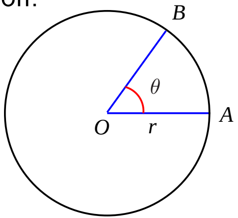
3. In particular, we have

$$360^\circ = 2\pi \text{ rad} \qquad 180^\circ = \pi \text{ rad} \qquad 45^\circ = \frac{\pi}{4} \text{ rad}$$

$$30^\circ = \frac{\pi}{6} \text{ rad} \qquad 60^\circ = \frac{\pi}{3} \text{ rad} \qquad -90^\circ = \frac{\pi}{2} \text{ rad}$$

Radian Measure of an Angle

Since the length and area of a circle of radius r are $2\pi r$ and πr^2, the arc length and area of a circular section subtended by an angle θ in radians can be determined according to the following proportion:



$$\frac{\text{circular sector area}}{\text{circle area}} = \frac{\theta}{2\pi} = \frac{\text{circular arc length}}{\text{circle length}}$$

$$\frac{\text{circular sector area}}{\pi r^2} = \frac{\theta}{2\pi} = \frac{\text{circular arc length}}{2\pi r}$$

and hence we have

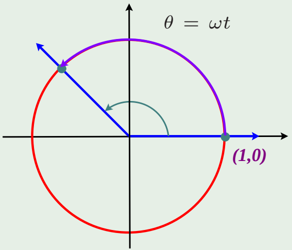
$$\text{circular sector area} = \frac{1}{2} r^2 \theta \quad \text{and} \quad \text{circular arc length} = r\theta$$

where θ is measured in radians, NOT degrees.

Radian Measure of an Angle

Example

If a particle is moving along a unit circle with *angular velocity* ω radians per second, then the angle subtended after t seconds is given by $\theta = \omega t$ radians, which is the distance traveled by the particle.



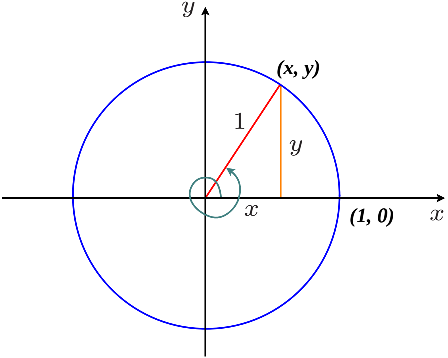
If the radius of the circle is R, then the distance traveled by the particle after t seconds is $R\omega t$.

Outline

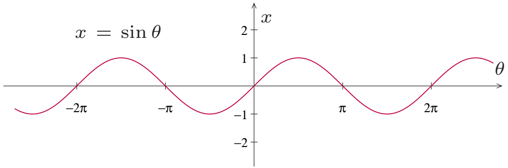
Sine and Cosine Functions

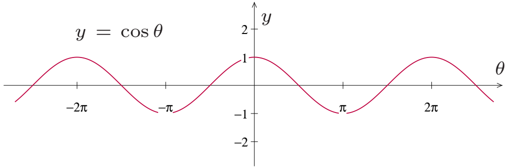
When a point originally at $(0, 1)$ moves along the unit circle through an angle of θ radians, the coordinates of the position (x, y) reached by the point depend on the value of θ, i.e., they are functions of θ:

$$y = \sin\theta \quad \text{and} \quad x = \cos\theta, \quad \text{where } \theta \in (-\infty, +\infty) \text{ and } x, y \in [-1, 1].$$



It is easy to plot the graphs of $x = \sin\theta$ and $y = \cos\theta$ from the geometry of the circle.

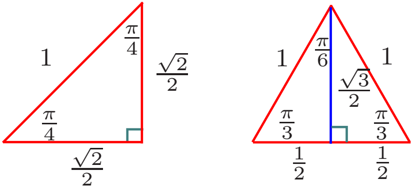




Since θ and $2\pi + \theta$ give you the same point on the unit circle, we have $\sin(\theta + 2\pi) = \sin\theta$ and $\cos(\theta + 2\pi) = \cos\theta$ i.e., both functions are periodic with period 2π.

Some Function Values of $\sin\theta$ and $\cos\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	-1

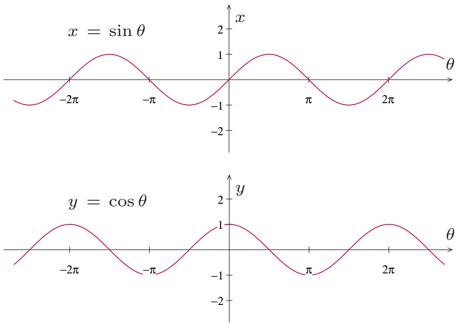


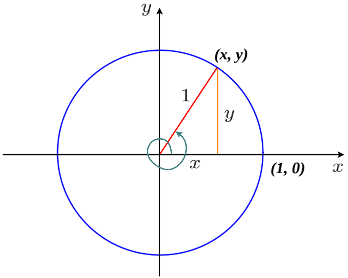
- $\sin\theta = 0$ if and only if $\theta = n\pi$ for some integer n. (points on the unit circle with zero y-coordinates are $(\pm 1, 0)$)
- $\cos\theta = 0$ if and only if $\theta = (2n+1)\frac{\pi}{2} = (n+\frac{1}{2})\pi$ for some integer n. (points on the unit circle with zero x-coordinates are $(0, \pm 1)$)

Properties of Sine and Cosine

Note that we have the following identities:

1. $\sin^2\theta + \cos^2\theta = 1$ (Pythagoras Theorem)
2. $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$ (graph shifting)
3. $\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$ (graph shifting)

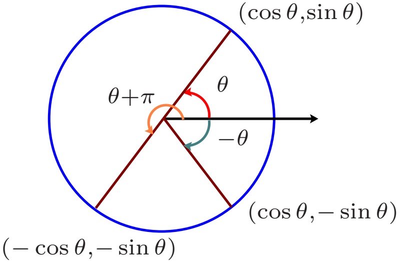




Properties of Sine and Cosine

Since θ and $-\theta$ put two points on unit circle symmetric with respect to x-axis and $\theta+\pi$ gives a point antipodal to that of θ, we have

$$\sin(-\theta) = -\sin\theta,$$
$$\cos(-\theta) = \cos\theta,$$
$$\sin(\theta+\pi) = -\sin\theta,$$
$$\cos(\theta+\pi) = -\cos\theta.$$



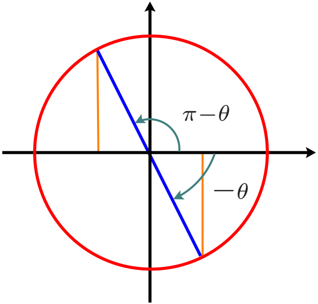
It is easy to see that $\sin\theta$ is an odd function and $\cos\theta$ is an even function.

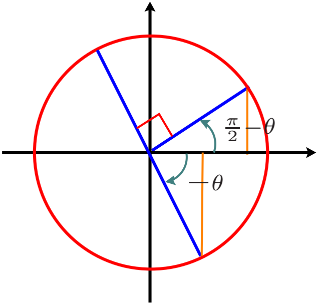
Properties of Sine and Cosine

By studying points on the unit circle given by the angles θ, $\frac{\pi}{2} - \theta$ and $\pi - \theta$, we have:

$$\sin(\pi - \theta) = \sin\theta, \qquad \sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta,$$

$$\cos(\pi - \theta) = -\cos\theta, \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta.$$

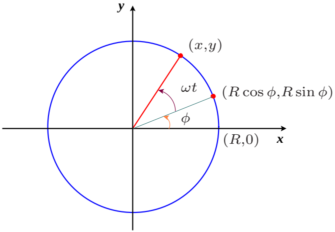




Properties of Sine and Cosine

1. If (x, y) is a point on the circle of radius R, with the equation $x^2 + y^2 = R^2$, we have by proportion that
$$x = R\cos\theta, \qquad y = R\sin\theta \ .$$
2. If the point is rotating around the circle with constant angular velocity ω from $(R, 0)$, then at time t, the x and y coordinates of the point are given by
$$x = R\cos(\omega t), \qquad y = R\sin(\omega t) \ .$$
3. If the initial position of the point is $(R\cos\phi, R\sin\phi)$ instead of $(R, 0)$, the coordinate functions of the point are given by
$$x = R\cos(\omega t + \phi), \qquad y = R\sin(\omega t + \phi) \ .$$

The rotation is
$$\begin{cases} \text{counterclockwise}, & \text{if } \omega > 0, \\ \text{clockwise}, & \text{if } \omega < 0. \end{cases}$$



Properties of Sine and Cosine

1. It is clear that functions $x = R\cos(\omega t + \phi)$ and $y = R\sin(\omega t + \phi)$ are periodic with period $2\pi/|\omega|$.
2. Such functions are often used in describing certain periodic oscillation motion, namely, *simple harmonic motion*. R is called the *amplitude*, $-\phi/\omega$ the *phase shift* and $\omega t + \phi$ the *phase* or *phase angle*.
3. The graphs of these functions can be easily found by performing suitable transformations on the graph of the sine or cosine function.