Calculus IB: Lecture 02

Luo Luo

Department of Mathematics, HKUST

http://luoluo.people.ust.hk/

Outline

Outline

What is a Function?

- A *function* f is a rule that assigns to each element x in a set D exactly one element in a set E, which is denoted by $f(x)$ and called the *function value of f at x*.
- The set D is called the *domain of f* and the set E is called the *codomain of f*.
- A function f with domain D and codomain E is usually denoted by $f : D \longrightarrow E$.
- We can think of a function $f : D \longrightarrow E$ as an input-output machine which produces a unique output value $f(x)$ in the codomain E for any given input value x taken from the domain D.
- By considering the set of all function values of f, we have the *range* of the function: $\text{range of } f = \{f(x) : x \text{ is in the domain } D\}$.
- Note that the range of a function $f : D \longrightarrow E$ may not be the whole codomain E. f is said to be *onto* or *surjective* if $E = \text{range of } f$.

What is a Function?

- In Math1013, the domain D and codomain E of a function f are usually certain sets of real numbers unless mentioned otherwise.
- In fact, E is most often taken as the set of all real numbers $\mathbb{R}$ when the function is given by a mathematical formula of the form $y = f(x)$; e.g., $y = f(x) = x^2 + x^3$, or just $y = x^2 + x^3$, while the codomain of the function is not explicitly mentioned.
- Given a function $y = f(x)$, the symbol x which represents numbers in the domain of f is called the *independent variable*, and the symbol y, which represents the function values in the range of f, is called the *dependent variable*.

Graph of a Function

The *graph* of a function $f : D \longrightarrow E$ is just the set of ordered pairs of numbers

$$\text{graph of } f = \{(x, f(x)) : x \text{ is a number in } D\}$$

which can be geometrically plotted as a set of coordinate points in the xy-plane, if the function f is not too complicated.

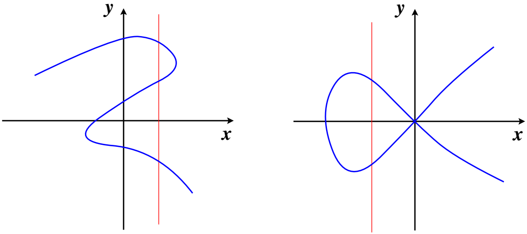
Vertical line test for the graph of a function

- The graph of any function f should intersect every vertical line at most *once* (since for any number c in the domain of f, only one function value $f(c)$ is assigned).
- Conversely, any set of points in the xy-plane passing this test can be used to defined a function graphically.

Graph of a Function

Vertical line test for the graph of a function

- The graph of any function f should intersect every vertical line at most once (since for any number c in the domain of f, only one function value $f(c)$ is assigned).
- Conversely, any set of points in the xy-plane passing this test can be used to defined a function graphically.



These curves cannot be the graph of any function, since they fail the vertical line test

What is a Function?

A function is usually used to relate two quantities, namely, to indicate how the value of a quantity (in the domain) determines uniquely the value of another quantity (in the codomain).

A function can be described in the following ways:

1. verbally; (by a description in words)
2. numerically; (by a table of function values, often partially)
3. visually; (by a graph)
4. by an explicit formula.

Some basic examples of functions

Example

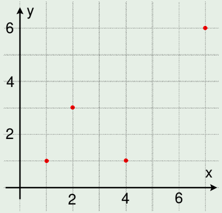
Consider a domain $D = \{1, 2, 4, 7\}$, and the codomain $\mathbb{R}$ which is the set of all real numbers. Let the rule of the function $f : D \longrightarrow \mathbb{R}$, which assigns function values to numbers in D, be defined explicitly by setting

$$f(1) = 1, \quad f(2) = 3, \quad f(4) = 1, \quad f(7) = 6.$$

The graph of f is obviously the set of four ordered pairs of numbers $\{(1, 1), (2, 3), (4, 1), (7, 6)\}$, which can also be described in terms of a simple table of function values, or a figure of four points in the xy-plane:

x	1	2	4	7
$f(x)$	1	3	1	6

$\overset{or}{\longleftrightarrow}$



Some basic examples of functions

Describing a function f by a complete table of function values, or a complete geometric graph, is not always feasible.

Especially when the domain D of f has too many numbers; e.g., what if domain D contains infinitely many numbers?

Some basic examples of functions

Example

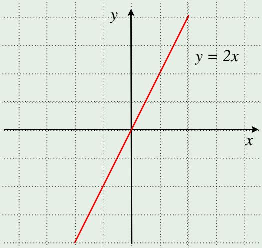
Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where the rule for assigning function values is defined by the following mathematical formula $f(x) = 2x$.

It is then easy to figure out any function value you want, e.g.,

$$f(2) = 2 \cdot 2 = 4, \quad f(21) = 2 \cdot 21 = 42, \qquad f(-3) = 2 \cdot (-3) = -6$$

However, it is obviously impossible to show a complete table of function values, as there are infinitely many real numbers.

This function is simple enough to sketch: a straight line in the xy-plane through the origin with slope 2.



Some basic examples of functions

Review the basic things about "*linear functions*", i.e., functions of the form $y = mx + c$, where $m \neq 0$ is a constant called the *slope* of the function, and c a constant called the *y-intercept*.

Exercise

1. How do you find the "*x-intercept*", "*y-intercept*" and *slope* of a linear function? For example, determine the intercepts of the linear function $f(x) = -2x + 3$ and sketch its graph.
2. Sketch the graph of a few more linear functions; e.g., $y = 3x + 5$, or $y = -2x - 6$.
3. What sorts of given conditions are sufficient for you to figure out the equation of a straight line? For example, what if (i) $(2, 3)$ is a point on the straight line, and 2 is the slope; or (ii) $(2, 3)$ and $(6, 9)$ are two points on the straight line?

Some basic examples of functions

Example

Let $C(w)$ be cost of mailing a *small letter* with weight w not exceeding 50g in Hong Kong. Obviously, C is a function with domain given by $\{w : 0 < w \leq 50\}$ (in grams). To come up with the exact cost formula for $C(w)$, you need to know the rules of the Hong Kong Post Office (`https://www.hongkongpost.hk`):

Weight not over	Cost ($)
30g	2
50g	3

Now, we can give an explicit mathematical description of C:

$$C(w) = \begin{cases} 2 & \text{if } 0 < w \leq 30 \\ 3 & \text{if } 30 < w \leq 50 \end{cases}$$

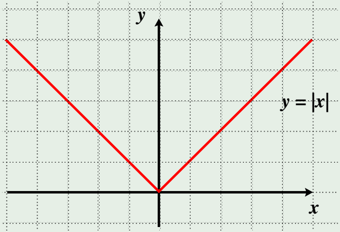
The range of the function C is obviously $\{2, 3\}$. See if you could sketch the graph of this *step function*, which is an example of *piece-wise constant functions*.

Some basic examples of functions

Example

The *absolute value function*, denoted by $f(x) = |x|$, is given by the following case by case formula:

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



$f(x) = |x|$ is an example of what people call *piecewise linear functions*. The domain of f is the set of all real numbers, and the range is the set of all non-negative real numbers.

Some basic examples of functions

Exercise

1. Sketch the graph of $f(x) = |2x - 4|$, which is another example of a piece-wise linear function.
2. Sketch the graph of the function ("*unit step function*")

$$y = u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

where c is some fixed constant.

3. Sketch the graph of the function $y = 3u_2(t) - 2u_4(t)$; and then use piece-wise defined formula to describe the function.

Some basic examples of functions

Example (real wold)

Let's just take the domain D to be an appropriate set of real numbers in each context below, and codomain $E = \mathbb{R}$ the set of all real numbers. All units are SI units when applicable.

numbers in D represent	function values in E represent
side length	area of the square with given side length
temperature in Celsius	same temperature in Fahrenheit

These functions above can be described more clearly by mathematical formulas if suitable symbols are introduced to denote the quantities involved.

numbers in D represent	function values in E represent
$x =$ side length	$A =$ area of the square...
$C =$ temperature in Celsius	$F =$ same temperature in Fahrenheit

$$A = x^2 \text{ (area of a square)}, \quad F = \frac{9}{5}C + 32 \text{ (unit conversion)}$$

Some basic examples of functions

Remark

Some domains of above functions are restricted to a certain range of values limited by physical restrictions.

1. *For the area function, we have* $D = \{x : x \geq 0\}$.
2. *The minimum temperature is taken as -273.15 on the Celsius scale.*

*It is important to understand the "*practical domain*" of a function in any modeling application, i.e., possible inputs of the function limited by the assumptions of the model, instead of just the "*natural domain*" of the function, i.e., where the formula makes sense mathematically.*

Outline

Some Elementary Functions

Following elementary mathematical functions you need to get familiar

- constant functions; e.g., 2, π, e.
- polynomial functions; e.g., $f(x) = x^3 + 2x^2 - 4x + 5$.
- rational functions; e.g., $f(x) = \dfrac{x^3 + 2x^2 - 4x + 5}{x^2 + 2x + 7}$.
- power functions; e.g., $f(x) = x^{3/2}$.
- exponential functions; e.g., $f(x) = 10^x$.
- logarithmic functions; e.g., $f(x) = \log_{10} x$.
- trigonometric functions; e.g., $\sin x$, $\cos x$, $\tan x$.
- inverse trigonometric functions; e.g., $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$.

Outline

Basic Operations: Sum, Product and Quotient

Given real-valued functions f and g, we can define new functions $f + g$ (*sum*), fg (*product*), and $\frac{f}{g}$ (*quotient*) simply by setting following rules:

$$(f + g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

as long as both function values, $f(x)$ and $g(x)$, are well-defined, and the corresponding arithmetic operations on them are valid.

However, we need to be careful with the domains of these functions.

Domains of sum, product and quotient

1. For either $(f+g)(x)$ or $(fg)(x)$, the input value x must be in both the domain of f and the domain of g in order to have well-defined function values to add or to multiply. Hence the domain of $f+g$, or fg, is

$$\{x : x \text{ is is the domain of } f \text{ and } x \text{ is also in the domain of } g\}$$

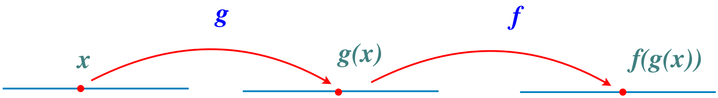
2. For $\dfrac{f(x)}{g(x)}$ to be well-defined, $f(x)$ and $g(x)$ have to be well-defined, and $g(x)$ has to be non-zero. Hence the domain of the function $\dfrac{f}{g}$ is

$$\{x : x \text{ is in the domain of } f, \text{ and } x \text{ is in the domain of } g, \text{ and } g(x) \neq 0\}$$

Basic Operations: Composition

One can also connect two "input-output machines" (functions) to form a new function, called the *composition* of f and g and denoted by the notation $f \circ g$, which is defined by

$$(f \circ g)(x) = f(g(x))$$



Obviously, we need $g(x)$ to be well-defined first, and then $g(x)$ to be in the domain of f in order to have a well-defined function value $f(g(x))$. Hence the domain of $f \circ g$ is given by

$$\begin{aligned} &\text{domain of } f \circ g \\ =&\{x : x \text{ is in the domain of } g \text{ and } g(x) \text{ is in the domain of } f\} \end{aligned}$$

Basic Operations: Composition

Some basic functions can simply be built by applying basic operations we mentioned to the constant functions and the linear function $f(x) = x$:

1. a constant function: $f(x) = 8$, $D = (-\infty, \infty)$
2. a linear function: $f(x) = 2x + 3$, $D = (-\infty, \infty)$
3. a quadratic function: $f(x) = 2x^2 - 4x + 8$ $D = (-\infty, \infty)$
4. a polynomial function of degree 5: $g(x) = 3x^5 + 5x^4$, $D = (-\infty, \infty)$.
5. a rational function: $r(x) = \dfrac{x^2 + 4x + 4}{x^2 - 1}$, we require $x \neq \pm 1$ to avoid a zero denominator which leads to $D = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

Polynomial Function and Rational Function

1. A *polynomial function of degree n* is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where, and $a_0, \ldots, a_n$ are some constants.

2. A *rational function* is the quotient of two polynomials, i.e., a function of the form

$$R(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$$

where n, m are non-negative integers, and $a_0, \ldots, a_n, b_0, \ldots, b_n$ are some constants with $a_n \neq 0$ and $b_m \neq 0$.

More Examples on Basic Operations of Functions

Example

Consider $f(x) = 2x - 1$, $g(x) = x^2 - 1$, then

$$(f+g)(x) = (2x-1) + (x^2-1) = 2x - x^2 - 2$$

$$(fg)(x) = (2x-1)(x^2-1) = 2x^3 - x^2 - 2x + 1$$

$$(5f)(x) = 5(2x-1) = 10x - 5$$

$$\left(\frac{f}{g}\right)(x) = \frac{2x-1}{x^2-1}$$

What are the domains of these functions?

Examples on Composition Operation

Example

Suppose $f(x) = 2x - 1$, $g(x) = x^2 - 1$ as above. Then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x^2 - 1) \\ &= 2(x^2 - 1) - 1 \\ &= 2x^2 - 3\end{aligned}$$

(Note that $f(\bigstar) = 2\bigstar - 1$)

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x - 1) \\ &= (2x - 1)^2 - 1 \\ &= 4x^2 - 4x\end{aligned}$$

(Note that $g(\bigstar) = \bigstar^2 - 1$)

Note that in general, $f \circ g$ and $g \circ f$ are not the same function.

Examples on Composition Operation

Example

Let $f(x) = \dfrac{1}{x}$. Find $f \circ f$.

We have

$$(f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} \stackrel{?}{=} x$$

Is the function $f \circ f$ the same as the function $h(x) = x$?

No! The domain of h is the set of all real numbers, but $x = 0$ is not in the domain of $f \circ f$, since $1/0$ is not a well-defined number.

If the answer to the function $f \circ f$ above is given in the "simplified form" of

$$(f \circ f)(x) = x,$$

it should be stated that there is actually a domain restriction $x \neq 0$.

Examples on Composition Operation

Example

Let $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{x+1}{x-2}$. Find $g \circ f$.

We have

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{\frac{1}{x}+1}{\frac{1}{x}-2}$$

with domain determined by the condition

$$x \neq 0 \quad \text{and} \quad \frac{1}{x} - 2 \neq 0$$

i.e., $x \neq 0, \dfrac{1}{2}$.

Examples on Composition Operation

Example

Let $f(x) = x^2 - 3$, $g(x) = \sqrt{x-1}$. Find (i) $f \circ g$ and (ii) $g \circ f$.

Note that the domain of f is $(-\infty, \infty)$, and the domain of g is $[1, \infty)$.

(i) $(f \circ g)(x) = f(g(x)) = f(\sqrt{x-1}) = (\sqrt{x-1})^2 - 3$, with domain given by $x \geq 1$. Similarly, $(f \circ g)(x)$ is not exactly the same as the function $h(x) = x - 4$ since their domains are different.

(ii) $(g \circ f)(x) = g(f(x)) = g(x^2 - 3) = \sqrt{(x^2-3)-1}$. By working with the sign line of $(x^2 - 3) - 1 = (x-2)(x+2)$, the domain of $g \circ f$ can be found as $(-\infty, -2] \cup [2, \infty)$ from the condition that $(x-2)(x+2) \geq 0$.

What about the domain of $\sqrt[n]{b}$ (n-th root of b)?

More discussion on $\sqrt[n]{b}$

1. For any positive even number n, the *radical expression* $\sqrt[n]{b}$ denotes the positive root of the equation $x^n = b$. For example, $\sqrt[4]{16} = 2$ since $2^4 = 16$. No *real* root exists if b is negative, e.g., $\sqrt[4]{-16}$ does not exist as a real number since $x^4 \geq 0 > -16$ for any real number x, i.e., $x^4 = -16$ has no real solution.
2. For any positive odd number n, the equation $x^n = b$ has a unique *real* root for any given real number b, which is also denoted by $\sqrt[n]{b}$. For examples, $\sqrt[3]{8} = 2$ since $2^3 = 8$, and $\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$.
3. Recall that a radical expression can also be expressed in terms of *exponent notation*; e.g., $\sqrt[n]{x} = x^{\frac{1}{n}}$ for any positive integer n. The relation between the *power function* $y = x^n$ and the *n-th root function* $y = \sqrt[n]{x}$ will be discussed in more detail later when we deal with the concept of *inverse function*.

More Discussion on Composition Operation

The composition of more than two functions can also be defined accordingly. For example, the composition $f \circ g \circ h$ is defined by

$$(f \circ g \circ h)(x) = f(g(h(x))) \ .$$

It is easy to see also that $f \circ g \circ h = f \circ (g \circ h) = (f \circ g) \circ h$. *What is the domain of this function*?

Exercise

Consider $f(x) = 2x - 1$, $g(x) = \dfrac{x^2+1}{x}$. Find the following functions, and determine their domains.

(a) $f(g(x))$ (b) $g(f(x))$ (c) $(f \circ g \circ f)(x) = f(g(f(x)))$

Outline

Even and Odd Functions

A function $y = f(x)$ is called an $\begin{cases} \text{even function if } f(-x) = f(x) \\ \text{odd function if } f(-x) = -f(x) \end{cases}$.

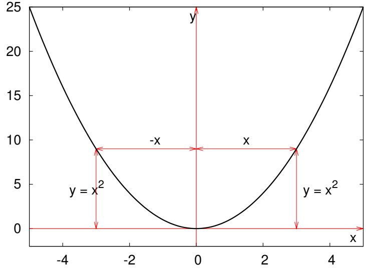
for all x in the domain of f.

Example

1. $y = x^2$ is an even function since $f(-x) = (-x)^2 = x^2 = f(x)$
2. $y = x^3$ is an odd function since $f(-x) = (-x)^3 = -x^3 = f(x)$
3. $y = |x|$ is an even function since $f(-x) = |-x| = |x| = f(x)$
4. $y = \dfrac{1}{x}$ is an odd function since $f(-x) = \dfrac{1}{-x} = -f(x)$ for $x \neq 0$

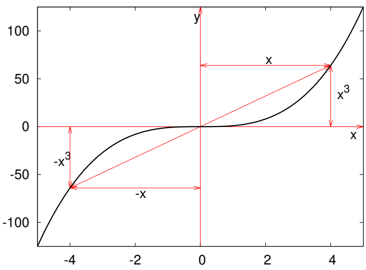
Even and Odd Functions

$(1) y = x^2$ is an even function



the graph of an even function is symmetric with respect to the y-axis (graph remains unchanged after reflection about y-axis)

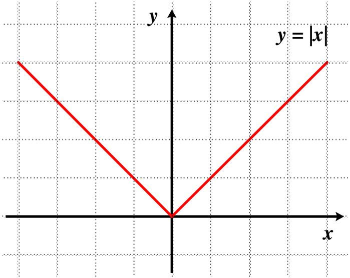
$(2) y = x^3$ is an odd function



the graph of an odd function is symmetric with respect to the origin (graph remains unchanged after rotation of 180 degrees about origin)

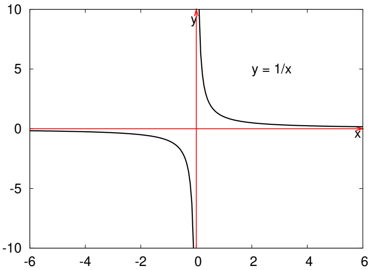
Even and Odd Functions

(1) $y = |x|$ is an even function



the graph of an even function is symmetric with respect to the y-axis

(2) $y = \frac{1}{x}$ is an odd function



the graph of an odd function is symmetric with respect to the origin

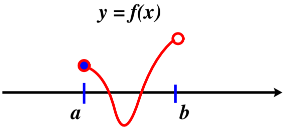
Periodic Functions

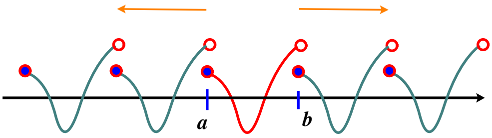
A function $f(x)$ is *periodic* if there is a number $T \neq 0$ such that $f(x+T) = f(x)$ for all x in the domain. The smallest such $T > 0$, if it exists, is called the *(fundamental) period* of the periodic function.

Note that a periodic function may has no (fundamental) period, why?

The graph of a periodic function does not change, if it is shifted to the left (or right), by a distance equal to an integral multiple of the period.

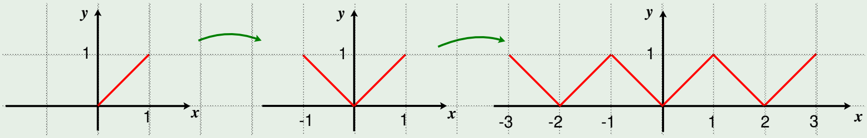
Any function f defined on the interval $[a, b)$ can be extended to a periodic function defined on the entire real line: keep shifting the graph by a distance of $b - a$.





Examples

Given a function $f(x) = x$ defined for $0 \le x \le 1$. Extend $f(x)$ to the whole real line as an even periodic function of period 2.



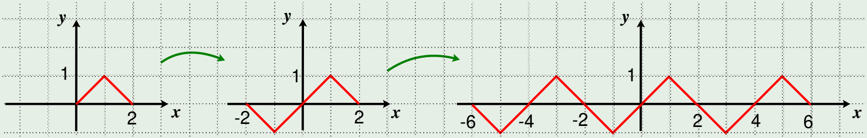
Examples of Periodic Functions

Example

Given a function

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ -x+2 & \text{if } 1 \le x \le 2 \end{cases}$$

defined on the interval $0 \le x \le 2$. Extend $f(x)$ to the whole real line as an odd periodic function of period 4.



Increasing and Decreasing Functions

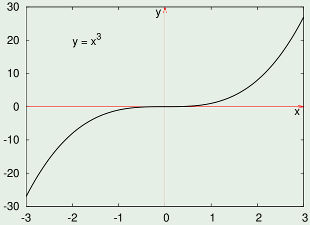
A function $y = f(x)$ is called

$$\begin{cases} \text{an } \textcolor{red}{\text{increasing function}} & \text{if } f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \\ \text{a } \textcolor{red}{\text{decreasing function}} & \text{if } f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \end{cases}$$

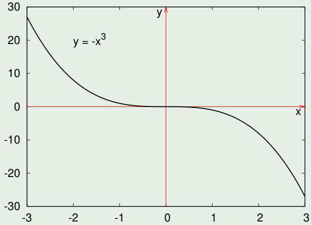
for all x_1, x_2 in the domain of f.

Example

increasing function: $y = x^3$



decreasing function: $y = -x^3$



The graph is rising/dropping when travelling along the positive x-direction. By arithmetic, $x_2^3 > x_1^3$ whenever $x_2 > x_1$, hence x^3 is an increasing function.

Outline

Transformations of Graphs

1. Graph of $y = f(x) + k$:
   $$\begin{cases} \text{upward shifting of the graph of } f \text{ by } k \text{ units if } k > 0 \\ \text{downward shifting of the graph of } f \text{ by } k \text{ units if } k < 0 \end{cases}$$
2. Graph of $y = f(x + k)$:
   $$\begin{cases} \text{shifting the graph of } f \text{ to the right by } |k| > 0 \text{ units if } k < 0 \\ \text{shifting the graph of } f \text{ to the left by } k \text{ units if } k > 0 \end{cases}$$
3. Graph of $y = -f(x)$: reflecting the graph of f across the x-axis.
4. Graph of $y = f(-x)$: reflecting the graph of f across the y-axis.
5. Graph of $y = kf(x)$, where $k > 0$:
   $$\begin{cases} \text{stretching the graph of } f \text{ in } y\text{-direction by a factor of } k \text{ if } k > 1 \\ \text{compressing the graph of } f \text{ in } y\text{-direction by a factor of } k \text{ if } 0 < k < 1 \end{cases}$$
6. Graph of $y = f(kx)$, where $k > 0$:
   $$\begin{cases} \text{compressing the graph of } f \text{ in } x\text{-direction by a factor of } k \text{ if } k > 1 \\ \text{stretching the graph of } f \text{ in } x\text{-direction by a factor of } k \text{ if } 0 < k < 1 \end{cases}$$

Example (completing the square)

Given the graph of $y = x^2$, sketch the graph of $y = -2(x+1)^2 - 2$ by using suitable transformations of graphs. Consider the following sequence of transformations:

$y = x^2$

$\downarrow$

$y = -x^2$

$\downarrow$

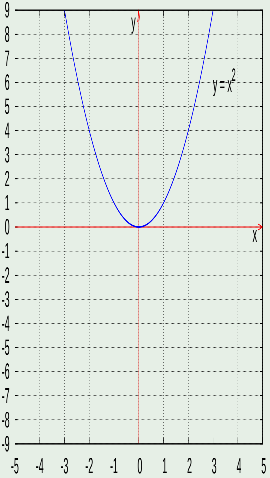
$y = -2x^2$

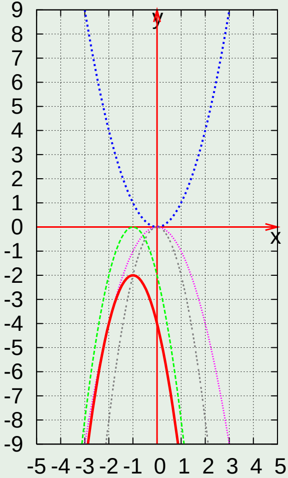
$\downarrow$

$y = -2(x+1)^2$

$\downarrow$

$y = -2(x+1)^2 - 2$





Example (completing the square, continued)

A quadratic polynomial $y = ax^2 + bx + c$, where $a \neq 0$, can be written as

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Thus the vertex of its graph is given by the coordinate point

$$\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right),$$

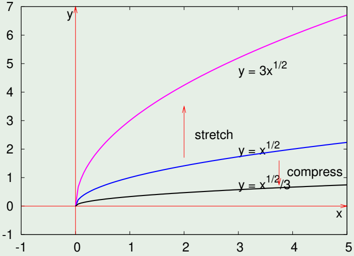
which is the lowest point on the graph if $a > 0$, and highest point on the graph if $a < 0$. The graph is symmetric with respect to the vertical line $x = -\frac{b}{2a}$, the *axis of symmetry*.

Examples of Transformations of Graphs

Example

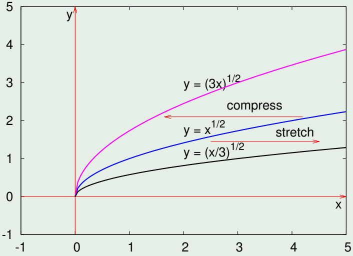
Consider the function defined by $f(x) = \sqrt{x}$. Compare the graph of $y = f(x)$ with the graphs of $y = 3f(x)$,$y = \frac{1}{3}f(x)$, $y = f(3x)$ and $y = f\left(\frac{x}{3}\right)$:

$3f(x) \longleftrightarrow f(x) \longleftrightarrow \frac{1}{3}f(x)$



$y = 3\sqrt{x}, y = \sqrt{x}, y = \frac{1}{3}\sqrt{x}$

$f(3x) \longleftrightarrow f(x) \longleftrightarrow f\left(\frac{x}{3}\right)$



$y = \sqrt{3x}, y = \sqrt{x}, y = \sqrt{\frac{x}{3}}$