Calculus IB: Lecture 01

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Outline

1. Course Overview
2. Sets and Intervals
3. Solving Inequalities
4. Absolute Value

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Why Calculus is Important?

Calculus is used in everywhere

- mathematics,
- physical science,
- computer science,
- statistics,
- engineering,
- economics,
- ......

Engineering would be almost impossible without calculus today.

I believe an understanding of calculus is never wasted.

Course Overview

Topics in single variable calculus

1. functions and graphs
2. limits of functions and continuity
3. derivatives and their applications
4. indefinite and definite integrals

Intended learning outcomes

1. develop basic computational skills in calculus
2. express quantitative relationships by the language of functions
3. apply calculus in modeling and solving real-world problems

Assessment Scheme and Resources

Percentage of coursework and examination

1. 25% by online homework (`https://www.classviva.org`)
2. no midterm exam
3. 75% by final exam

Recommended reading:

1. Jishan Hu, Weiping Li and Yueping Wu. "Calculus for scientists and engineers with MATLAB".
2. James Stewart. "Single variable calculus: Early transcendentals". Cengage Learning, 2015.

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Notations of Sets

A *set* is a well-defined collection of distinct elements.

1. We can list all elements: e.g., the expression $\{2, 5, 7\}$ means a set consisting of three numbers: 2, 5 and 7.
2. Capital letters are often used to denote a set; e.g., $A = \{2, 5, 7\}$, where 2, 5, 7 are called the elements of the set A.
3. The set of all real numbers is often denoted by the symbol $\mathbb{R}$.
4. The set of all integer is often denoted by the symbol $\mathbb{Z}$.
5. We use $\{x : P(x)\}$ to denote the set which is consisted of all elements x satisfying the description $P(x)$.

Notations of Sets

Examples of notation $\{x : P(x)\}$

1. $\{x : (x-2)(x-3) = 0\}$ is actually a set of two numbers: 2, 3
2. $\{x : (x-2)(x-3) > 0\}$ is the solution set of the inequality: $(x-2)(x-3) > 0$
3. $\{x : x \text{ is the square of an integer}\}$ is the set of 0, 1, 4, 9, 16, 25...

Sets can be consisting of things other than numbers in general; e.g.,

$$\{x : x \text{ is a HKUST student}\}$$

Notations of Intervals

Infinity, denoted by ∞, represents something that is larger than any real number. Similarly, we use $-\infty$ to represent *negative infinity* that is smaller than any real number.

An *interval* is a set of real numbers that contains all real numbers lying between any two endpoints.

1. An endpoint could be a real number, infinity or negative infinity.
2. What is "between"?

Notations of Intervals

Let a and b be two real numbers. We define different classes of interval as follows.

Open Intervals	Closed Intervals
$(a, b) = \{x : a < x < b\}$	$[a, b] = \{x : a \leq x \leq b\}$
$(-\infty, a) = \{x : x < a\}$	$(-\infty, a] = \{x : x \leq a\}$
$(a, \infty) = \{x : x > a\}$	$[a, \infty) = \{x : x \geq a\}$

Half Open Half Closed Intervals
$[a, b) = \{x : a \leq x < b\}$
$(a, b] = \{x : a < x \leq b\}$

The interval $(-\infty, \infty)$ formed by all real numbers, that is $\mathbb{R} = (-\infty, \infty)$, which is considered as both open and closed.

The interval $[a, b] = (a, b) = [a, b) = (a, b] = (a, a) = [a, a) = (a, a]$ contains nothing when $a > b$. We call it empty set, denoted by $\emptyset$ or $\{\}$.

Basic Operations on Sets

Given two sets of real numbers A and B, the *intersection* $A \cap B$ and the *union* $A \cup B$ mean respectively the following:

$$A \cap B = \{x : x \text{ is a number in both } A \text{ and } B\}$$
$$A \cup B = \{x : x \text{ is a number either in } A \text{ or in } B\}$$

For examples,

$$\{1,2,3,4\} \cap \{3,4,9\} = \{3,4\}$$
$$\{1,2,3,4\} \cup \{3,4,9\} = \{1,2,3,4,9\}$$
$$(2,7) \cap [3,10) = \{x : 2 < x < 7 \text{ and } 3 \leq x < 10\} = [3,7)$$
$$(2,7) \cup (3,10) = \{x : 2 < x < 7 \text{ or } 3 < x < 10\} = (2,10)$$

The union of two intervals is not always an interval:

$$(-2,0) \cup [3,8) = \{x : -2 < x < 0 \text{ or } 3 \leq x < 8\}$$

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Solving Inequalities

Basic operations on inequalities: for any real numbers a, b, and c,

1. if $a < b$, then $a + c < b + c$;
2. if $a < b$, then $a - c < b - c$;
3. if $a < b$ and $c > 0$, then $ac < bc$;
4. if $a < b$ and $c < 0$, then $ac > bc$;

Watch out when multiplying a negative number c on $a < b$, the result is $ac > bc$, rather than $ac < bc$!

For example: $2 < 3$ leads to $2 \cdot (-4) > 3 \cdot (-4)$

Example

Solve the following inequalities: $4x - 3 < 2x + 5$

Solution

We apply basic operations on inequalities:

$$\begin{aligned} 4x - 3 &< 2x + 5 \\ 4x - 3 + (3 - 2x) &< 2x + 5 + (3 - 2x) \\ 2x &< 8 \\ x &< 4. \end{aligned}$$

Using interval notation, the solution of the inequality is $(-\infty, 4)$.

Solving Inequalities

Example

Solve the following inequalities $-\frac{2x}{3} < x + 4$.

Solution

We can solve it as follow:

$$
\begin{aligned}
-\frac{2x}{3} - x &< 4 \\
\frac{-5x}{3} &< 4 \\
\left(-\frac{3}{5}\right)\left(-\frac{5x}{3}\right) &> \left(-\frac{3}{5}\right) \cdot 4 \\
x &> -\frac{12}{5}
\end{aligned}
$$

Using interval notation, the solution of the inequality is: $(-\frac{12}{5}, \infty)$.

Solving Inequalities

Example

Solve the inequality $\dfrac{4}{2x-3} \leq 2$.

If you multiply $2x-3$ to both sides of the inequality, it is not clear how the inequality is changed since $2x-3$ may or may not be positive.

Solution

We have

$$\frac{4}{2x-3} - 2 \leq 0 \iff \frac{4}{2x-3} - \frac{2(2x-3)}{2x-3} \leq 0$$
$$\iff \frac{-4x+10}{2x-3} \leq 0.$$

The solution of the inequality is $x < \frac{3}{2}$ or $x \geq \frac{5}{2}$. Using interval notation, the solution is: $(-\infty, \frac{3}{2}) \cup [\frac{5}{2}, \infty)$.

Example

Solve the inequality $\dfrac{4}{2x-3} \leq 2.$

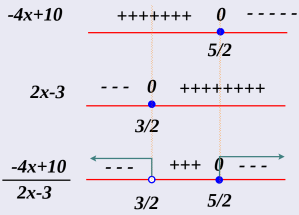
Solution

Why $\frac{-4x+10}{2x-3} \leq 0$ *leads to* $(-\infty, \frac{3}{2}) \cup [\frac{5}{2}, \infty)$*?*

x	$x < \frac{5}{2}$	$x = \frac{5}{2}$	$x > \frac{5}{2}$
$-4x+10$	$+ve$	0	$-ve$

x	$x < \frac{3}{2}$	$x = \frac{3}{2}$	$x > \frac{3}{2}$
$2x-3$	$-ve$	0	$+ve$

x	$x < \frac{3}{2}$	$x = \frac{3}{2}$	$\frac{3}{2} < x < \frac{5}{2}$	$x = \frac{5}{2}$	$x > \frac{5}{2}$
$\frac{-4x+10}{2x-3}$	$-ve$	undefined	$+ve$	0	$-ve$



Exercise

Solve the inequality $\dfrac{(x-2)(x-5)}{(x+2)(x-8)} \geq 0$.

Hint: There are four numbers $2, 5, -2$ and 8 divide the real line into five disjoint open intervals. We can do the sign checking for each of these intervals.

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Absolute Value

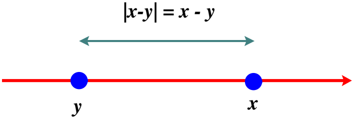
The absolute value of a real number x, denoted by $|x|$, is defined by

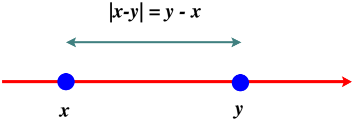
$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

For example, $|5| = 5$, and $|-5| = -(-5) = 5$. Similarly,

$$|x - y| = \begin{cases} x - y & \text{if } x \geq y, \\ y - x & \text{if } x < y. \end{cases}$$

The value of $|x - y|$ can also be seen as the distance between the numbers x and y on the real line.





Absolute Value

No matter what a mathematical expression $\blacksquare$, we have

$$|\blacksquare| = \begin{cases} \blacksquare & \text{if } \blacksquare \geq 0, \\ -\blacksquare & \text{if } \blacksquare < 0. \end{cases}$$

Note also that for any positive real number k, we have

1. $|\blacksquare| < k \iff -k < \blacksquare < k$
2. $|\blacksquare| > k \iff \blacksquare < -k$ or $\blacksquare > k$

Example

The equation $|2x - 5| = 3$ simply means $2x - 5 = 3$ or $2x - 5 = -3$, that is $x = 4$ or $x = 1$.

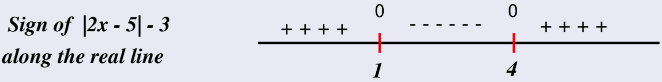
Example

The inequality $|2x-5|<3$ means

$$\begin{aligned}|2x-5|<3 &\iff -3<2x-5<3\\ &\iff 2<2x<8\\ &\iff 1<x<4\end{aligned}$$

Remark

The solution of $|2x-5|=3$ is $x=1$ or $x=4$ lead to we can solve $|2x-5|<3$ by sign checking for $|2x-5|-3$ along the real line, which is divided by 1 and 4 into three intervals $x<1$, $1<x<4$, and $x>4$:



Equations or Inequalities Involving Absolute Values

Example

Solving the inequality $\left|3-\frac{5}{x}\right|<1$ (Recall that $|\star|<1 \Leftrightarrow -1<\star<1$)

Solution (inequality approach)

$$-1<3-\frac{5}{x}<1 \Longleftrightarrow -1<\frac{3x-5}{x}<1$$

$$0<1+\frac{3x-5}{x} \quad \textit{and} \quad \frac{3x-5}{x}-1<0$$

$$0<\frac{4x-5}{x} \quad \textit{and} \quad \frac{2x-5}{x}<0$$

$$\left(x<0 \textit{ or } x>\frac{5}{4}\right) \quad \textit{and} \quad 0<x<\frac{5}{2}$$

$$\textit{i.e.,} \quad \frac{5}{4}<x<\frac{5}{2}$$

Equations or Inequalities Involving Absolute Values

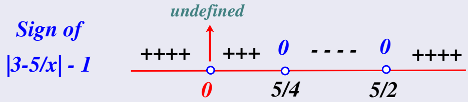
Example

Solving the inequality $\left|3 - \frac{5}{x}\right| < 1$ (Recall that $|\star| < 1 \Leftrightarrow -1 < \star < 1$)

Solution (equation approach)

The solution of $\left|3 - \dfrac{5}{x}\right| = 1$ *is either* $3 - \dfrac{5}{x} = -1$ *or* $3 - \dfrac{5}{x} = 1$,

that is $x = \dfrac{5}{4}$ *or* $x = \dfrac{5}{2}$. *Check the sign of* $\left|3 - \dfrac{5}{x}\right| - 1$*:*



$\left(e.g., \text{ check } \left|3 - \dfrac{5}{x}\right| \text{ at } x = -1, 1, 2, 3.\right)$

Some Exercises

Find the solution of the inequality

1. $\left|2x-5\right| \geq 3$
2. $\left|3-\dfrac{5}{x}\right| \geq 1$
3. $|x-1|+|x-3|<4$ (a harder one!)

Some Basic Properties of Absolute Values

Some Properties of Absolute Values:

1. $|-x| = |x|$
2. $|xy| = |x||y|$
3. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, where $y \neq 0$
4. $|x+y| \leq |x| + |y|$ (triangle inequality)

where equality holds if and only if x, y are of the same sign (equivalently $ab > 0$), or one of them is 0.

The Proof of Triangle Inequality

Why $|x+y| \leq |x|+|y|$ holds?

Proof.

It follows easily from

$$\begin{aligned} |x+y|^2 &= (x+y)^2 = x^2+2xy+y^2 \\ &= |x|^2+2xy+|y|^2 \\ &\leq |x|^2+2|x||y|+|x|^2 = (|x|+|y|)^2 \\ |x+y| &\leq |x|+|y| \end{aligned}$$

where equality holds if and only if $xy = |xy|$, equivalently, $xy \geq 0$. $\square$