Calculus IB: Lecture 01

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Outline

Outline

Why Calculus is Important?

Calculus is used in everywhere

- mathematics,
- physical science,
- computer science,
- statistics,
- engineering,
- economics,
- ......

Engineering would be almost impossible without calculus today.

I believe an understanding of calculus is never wasted.

Course Overview

Topics in single variable calculus

1. functions and graphs
2. limits of functions and continuity
3. derivatives and their applications
4. indefinite and definite integrals

Intended learning outcomes

1. develop basic computational skills in calculus
2. express quantitative relationships by the language of functions
3. apply calculus in modeling and solving real-world problems

Assessment Scheme and Resources

Percentage of coursework and examination

1. 25% by online homework (`https://www.classviva.org`)
2. no midterm exam
3. 75% by final exam

Recommended reading:

1. Jishan Hu, Weiping Li and Yueping Wu. "Calculus for scientists and engineers with MATLAB".
2. James Stewart. "Single variable calculus: Early transcendentals". Cengage Learning, 2015.

Outline

Notations of Sets

A *set* is a well-defined collection of distinct elements.

1. We can list all elements: e.g., the expression $\{2, 5, 7\}$ means a set consisting of three numbers: 2, 5 and 7.
2. Capital letters are often used to denote a set; e.g., $A = \{2, 5, 7\}$, where 2, 5, 7 are called the elements of the set A.
3. The set of all real numbers is often denoted by the symbol $\mathbb{R}$.
4. The set of all integer is often denoted by the symbol $\mathbb{Z}$.
5. We use $\{x : P(x)\}$ to denote the set which is consisted of all elements x satisfying the description $P(x)$.

Examples of notation $\{x : P(x)\}$

1. $\{x : (x-2)(x-3) = 0\}$ is actually a set of two numbers: 2, 3
2. $\{x : (x-2)(x-3) > 0\}$ is the solution set of the inequality: $(x-2)(x-3) > 0$
3. $\{x : x \text{ is the square of an integer}\}$ is the set of 0, 1, 4, 9, 16, 25...

Sets can be consisting of things other than numbers in general; e.g.,

$$\{x : x \text{ is a HKUST student}\}$$

Notations of Intervals

Infinity, denoted by ∞, represents something that is larger than any real number. Similarly, we use $-\infty$ to represent *negative infinity* that is smaller than any real number.

An *interval* is a set of real numbers that contains all real numbers lying between any two endpoints.

1. An endpoint could be a real number, infinity or negative infinity.
2. What is "between"?

Notations of Intervals

Let a and b be two real numbers. We define different classes of interval as follows.

Open Intervals	Closed Intervals
$(a, b) = \{x : a < x < b\}$	$[a, b] = \{x : a \leq x \leq b\}$
$(-\infty, a) = \{x : x < a\}$	$(-\infty, a] = \{x : x \leq a\}$
$(a, \infty) = \{x : x > a\}$	$[a, \infty) = \{x : x \geq a\}$

Half Open Half Closed Intervals
$[a, b) = \{x : a \leq x < b\}$
$(a, b] = \{x : a < x \leq b\}$

The interval $(-\infty, \infty)$ formed by all real numbers, that is $\mathbb{R} = (-\infty, \infty)$, which is considered as both open and closed.

The interval $[a, b] = (a, b) = [a, b) = (a, b] = (a, a) = [a, a) = (a, a]$ contains nothing when $a > b$. We call it empty set, denoted by $\emptyset$ or $\{\}$.

Basic Operations on Sets

Given two sets of real numbers A and B, the *intersection* $A \cap B$ and the *union* $A \cup B$ mean respectively the following:

$$A \cap B = \{x : x \text{ is a number in both } A \text{ and } B\}$$
$$A \cup B = \{x : x \text{ is a number either in } A \text{ or in } B\}$$

For examples,

$$\{1,2,3,4\} \cap \{3,4,9\} = \{3,4\}$$
$$\{1,2,3,4\} \cup \{3,4,9\} = \{1,2,3,4,9\}$$
$$(2,7) \cap [3,10) = \{x : 2 < x < 7 \text{ and } 3 \leq x < 10\} = [3,7)$$
$$(2,7) \cup (3,10) = \{x : 2 < x < 7 \text{ or } 3 < x < 10\} = (2,10)$$

The union of two intervals is not always an interval:

$$(-2,0) \cup [3,8) = \{x : -2 < x < 0 \text{ or } 3 \leq x < 8\}$$

Outline

Solving Inequalities

Basic operations on inequalities: for any real numbers a, b, and c,

1. if $a < b$, then $a + c < b + c$;
2. if $a < b$, then $a - c < b - c$;
3. if $a < b$ and $c > 0$, then $ac < bc$;
4. if $a < b$ and $c < 0$, then $ac > bc$;

Watch out when multiplying a negative number c on $a < b$, the result is $ac > bc$, rather than $ac < bc$!

For example: $2 < 3$ leads to $2 \cdot (-4) > 3 \cdot (-4)$

Example

Solve the following inequalities: $4x - 3 < 2x + 5$

Solution

We apply basic operations on inequalities:

$$\begin{aligned} 4x - 3 &< 2x + 5 \\ 4x - 3 + (3 - 2x) &< 2x + 5 + (3 - 2x) \\ 2x &< 8 \\ x &< 4. \end{aligned}$$

Using interval notation, the solution of the inequality is $(-\infty, 4)$.

Solving Inequalities

Example

Solve the following inequalities $-\frac{2x}{3} < x + 4$.

Solution

We can solve it as follow:

$$
\begin{aligned}
-\frac{2x}{3} - x &< 4 \\
\frac{-5x}{3} &< 4 \\
\left(-\frac{3}{5}\right)\left(-\frac{5x}{3}\right) &> \left(-\frac{3}{5}\right) \cdot 4 \\
x &> -\frac{12}{5}
\end{aligned}
$$

Using interval notation, the solution of the inequality is: $(-\frac{12}{5}, \infty)$.

Solving Inequalities

Example

Solve the inequality $\dfrac{4}{2x-3} \leq 2$.

If you multiply $2x-3$ to both sides of the inequality, it is not clear how the inequality is changed since $2x-3$ may or may not be positive.

Solution

We have

$$\frac{4}{2x-3} - 2 \leq 0 \iff \frac{4}{2x-3} - \frac{2(2x-3)}{2x-3} \leq 0$$
$$\iff \frac{-4x+10}{2x-3} \leq 0.$$

The solution of the inequality is $x < \frac{3}{2}$ or $x \geq \frac{5}{2}$. Using interval notation, the solution is: $(-\infty, \frac{3}{2}) \cup [\frac{5}{2}, \infty)$.

Solving Inequalities

Example

Solve the inequality $\dfrac{4}{2x-3} \leq 2$.

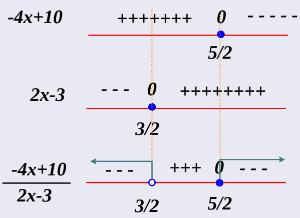
Solution

Why $\frac{-4x+10}{2x-3} \leq 0$ *leads to* $(-\infty, \frac{3}{2}) \cup [\frac{5}{2}, \infty)$*?*

x	$x < \frac{5}{2}$	$x = \frac{5}{2}$	$x > \frac{5}{2}$
$-4x+10$	$+ve$	0	$-ve$

x	$x < \frac{3}{2}$	$x = \frac{3}{2}$	$x > \frac{3}{2}$
$2x-3$	$-ve$	0	$+ve$

x	$x < \frac{3}{2}$	$x = \frac{3}{2}$	$\frac{3}{2} < x < \frac{5}{2}$	$x = \frac{5}{2}$	$x > \frac{5}{2}$
$\dfrac{-4x+10}{2x-3}$	$-ve$	undefined	$+ve$	0	$-ve$



Solving Inequalities

Exercise

Solve the inequality $\dfrac{(x-2)(x-5)}{(x+2)(x-8)} \geq 0$.

Hint: There are four numbers $2, 5, -2$ and 8 divide the real line into five disjoint open intervals. We can do the sign checking for each of these intervals.

Outline

Absolute Value

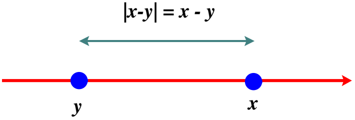
The absolute value of a real number x, denoted by $|x|$, is defined by

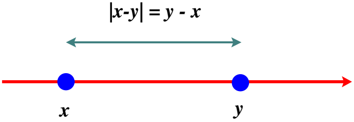
$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

For example, $|5| = 5$, and $|-5| = -(-5) = 5$. Similarly,

$$|x - y| = \begin{cases} x - y & \text{if } x \geq y, \\ y - x & \text{if } x < y. \end{cases}$$

The value of $|x - y|$ can also be seen as the distance between the numbers x and y on the real line.





Absolute Value

No matter what a mathematical expression $\blacksquare$, we have

$$|\blacksquare| = \begin{cases} \blacksquare & \text{if } \blacksquare \geq 0, \\ -\blacksquare & \text{if } \blacksquare < 0. \end{cases}$$

Note also that for any positive real number k, we have

1. $|\blacksquare| < k \iff -k < \blacksquare < k$
2. $|\blacksquare| > k \iff \blacksquare < -k$ or $\blacksquare > k$

Example

The equation $|2x-5| = 3$ simply means $2x - 5 = 3$ or $2x - 5 = -3$, that is $x = 4$ or $x = 1$.

Equations or Inequalities Involving Absolute Values

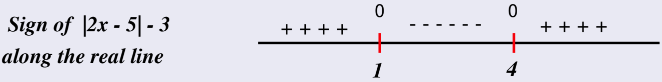
Example

The inequality $|2x-5|<3$ means

$$\begin{aligned} |2x-5|<3 &\iff -3<2x-5<3 \\ &\iff 2<2x<8 \\ &\iff 1<x<4 \end{aligned}$$

Remark

The solution of $|2x-5|=3$ is $x=1$ or $x=4$ lead to we can solve $|2x-5|<3$ by sign checking for $|2x-5|-3$ along the real line, which is divided by 1 and 4 into three intervals $x<1$, $1<x<4$, and $x>4$:



Equations or Inequalities Involving Absolute Values

Example

Solving the inequality $\left|3 - \frac{5}{x}\right| < 1$ (Recall that $|\star| < 1 \Leftrightarrow -1 < \star < 1$)

Solution (inequality approach)

$$-1 < 3 - \frac{5}{x} < 1 \iff -1 < \frac{3x-5}{x} < 1$$

$$0 < 1 + \frac{3x-5}{x} \quad \textit{and} \quad \frac{3x-5}{x} - 1 < 0$$

$$0 < \frac{4x-5}{x} \quad \textit{and} \quad \frac{2x-5}{x} < 0$$

$$\left(x < 0 \textit{ or } x > \frac{5}{4}\right) \quad \textit{and} \quad 0 < x < \frac{5}{2}$$

$$\textit{i.e.,} \;\; \frac{5}{4} < x < \frac{5}{2}$$

Equations or Inequalities Involving Absolute Values

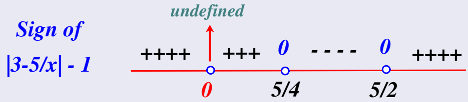
Example

Solving the inequality $\left|3 - \frac{5}{x}\right| < 1$ (Recall that $|\star| < 1 \Leftrightarrow -1 < \star < 1$)

Solution (equation approach)

The solution of $\left|3 - \dfrac{5}{x}\right| = 1$ *is either* $3 - \dfrac{5}{x} = -1$ *or* $3 - \dfrac{5}{x} = 1$,

that is $x = \dfrac{5}{4}$ *or* $x = \dfrac{5}{2}$. *Check the sign of* $\left|3 - \dfrac{5}{x}\right| - 1$:



$\left(e.g., \text{ check } \left|3 - \dfrac{5}{x}\right| \text{ at } x = -1, 1, 2, 3.\right)$

Equations or Inequalities Involving Absolute Values

Some Exercises

Find the solution of the inequality

1. $\left|2x-5\right| \geq 3$
2. $\left|3-\dfrac{5}{x}\right| \geq 1$
3. $|x-1|+|x-3|<4$ (a harder one!)

Some Basic Properties of Absolute Values

Some Properties of Absolute Values:

1. $|-x| = |x|$
2. $|xy| = |x||y|$
3. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, where $y \neq 0$
4. $|x+y| \leq |x| + |y|$ (triangle inequality)

where equality holds if and only if x, y are of the same sign (equivalently $ab > 0$), or one of them is 0.

The Proof of Triangle Inequality

Why $|x+y| \leq |x| + |y|$ holds?

Proof.

It follows easily from

$$\begin{aligned} |x+y|^2 &= (x+y)^2 = x^2 + 2xy + y^2 \\ &= |x|^2 + 2xy + |y|^2 \\ &\leq |x|^2 + 2|x||y| + |x|^2 = (|x| + |y|)^2 \\ |x+y| &\leq |x| + |y| \end{aligned}$$

where equality holds if and only if $xy = |xy|$, equivalently, $xy \geq 0$. □

Calculus IB: Lecture 02

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Outline

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What is a Function?

- A *function* f is a rule that assigns to each element x in a set D exactly one element in a set E, which is denoted by $f(x)$ and called the *function value of f at x*.
- The set D is called the *domain of f* and the set E is called the *codomain of f*.
- A function f with domain D and codomain E is usually denoted by $f : D \longrightarrow E$.
- We can think of a function $f : D \longrightarrow E$ as an input-output machine which produces a unique output value $f(x)$ in the codomain E for any given input value x taken from the domain D.
- By considering the set of all function values of f, we have the *range* of the function: $\text{range of } f = \{f(x) : x \text{ is in the domain } D\}$.
- Note that the range of a function $f : D \longrightarrow E$ may not be the whole codomain E. f is said to be *onto* or *surjective* if $E = \text{range of } f$.

What is a Function?

- In Math1013, the domain D and codomain E of a function f are usually certain sets of real numbers unless mentioned otherwise.
- In fact, E is most often taken as the set of all real numbers $\mathbb{R}$ when the function is given by a mathematical formula of the form $y = f(x)$; e.g., $y = f(x) = x^2 + x^3$, or just $y = x^2 + x^3$, while the codomain of the function is not explicitly mentioned.
- Given a function $y = f(x)$, the symbol x which represents numbers in the domain of f is called the *independent variable*, and the symbol y, which represents the function values in the range of f, is called the *dependent variable*.

Graph of a Function

The *graph* of a function $f : D \longrightarrow E$ is just the set of ordered pairs of numbers

$$\text{graph of } f = \{(x, f(x)) : x \text{ is a number in } D\}$$

which can be geometrically plotted as a set of coordinate points in the xy-plane, if the function f is not too complicated.

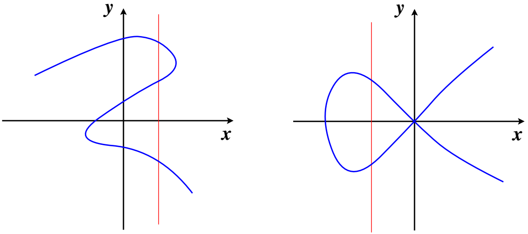
Vertical line test for the graph of a function

- The graph of any function f should intersect every vertical line at most *once* (since for any number c in the domain of f, only one function value $f(c)$ is assigned).
- Conversely, any set of points in the xy-plane passing this test can be used to defined a function graphically.

Graph of a Function

Vertical line test for the graph of a function

- The graph of any function f should intersect every vertical line at most once (since for any number c in the domain of f, only one function value $f(c)$ is assigned).
- Conversely, any set of points in the xy-plane passing this test can be used to defined a function graphically.



These curves cannot be the graph of any function, since they fail the vertical line test

What is a Function?

A function is usually used to relate two quantities, namely, to indicate how the value of a quantity (in the domain) determines uniquely the value of another quantity (in the codomain).

A function can be described in the following ways:

1. verbally; (by a description in words)
2. numerically; (by a table of function values, often partially)
3. visually; (by a graph)
4. by an explicit formula.

Some basic examples of functions

Example

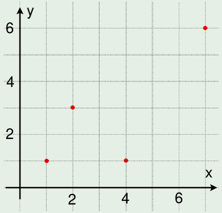
Consider a domain $D = \{1, 2, 4, 7\}$, and the codomain $\mathbb{R}$ which is the set of all real numbers. Let the rule of the function $f : D \longrightarrow \mathbb{R}$, which assigns function values to numbers in D, be defined explicitly by setting

$$f(1) = 1, \quad f(2) = 3, \quad f(4) = 1, \quad f(7) = 6.$$

The graph of f is obviously the set of four ordered pairs of numbers $\{(1,1), (2,3), (4,1), (7,6)\}$, which can also be described in terms of a simple table of function values, or a figure of four points in the xy-plane:

x	1	2	4	7
$f(x)$	1	3	1	6

$\overset{or}{\longleftrightarrow}$



Some basic examples of functions

Describing a function f by a complete table of function values, or a complete geometric graph, is not always feasible.

Especially when the domain D of f has too many numbers; e.g., what if domain D contains infinitely many numbers?

Some basic examples of functions

Example

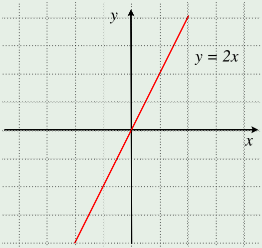
Consider the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ where the rule for assigning function values is defined by the following mathematical formula $f(x) = 2x$.

It is then easy to figure out any function value you want, e.g.,

$$f(2) = 2 \cdot 2 = 4, \quad f(21) = 2 \cdot 21 = 42, \qquad f(-3) = 2 \cdot (-3) = -6$$

However, it is obviously impossible to show a complete table of function values, as there are infinitely many real numbers.

This function is simple enough to sketch: a straight line in the xy-plane through the origin with slope 2.



Some basic examples of functions

Review the basic things about "*linear functions*", i.e., functions of the form $y = mx + c$, where $m \neq 0$ is a constant called the *slope* of the function, and c a constant called the *y-intercept*.

Exercise

1. How do you find the "*x-intercept*", "*y-intercept*" and *slope* of a linear function? For example, determine the intercepts of the linear function $f(x) = -2x + 3$ and sketch its graph.
2. Sketch the graph of a few more linear functions; e.g., $y = 3x + 5$, or $y = -2x - 6$.
3. What sorts of given conditions are sufficient for you to figure out the equation of a straight line? For example, what if (i) $(2, 3)$ is a point on the straight line, and 2 is the slope; or (ii) $(2, 3)$ and $(6, 9)$ are two points on the straight line?

Some basic examples of functions

Example

Let $C(w)$ be cost of mailing a *small letter* with weight w not exceeding 50g in Hong Kong. Obviously, C is a function with domain given by $\{w : 0 < w \leq 50\}$ (in grams). To come up with the exact cost formula for $C(w)$, you need to know the rules of the Hong Kong Post Office (`https://www.hongkongpost.hk`):

Weight not over	Cost ($)
30g	2
50g	3

Now, we can give an explicit mathematical description of C:

$$C(w) = \begin{cases} 2 & \text{if } 0 < w \leq 30 \\ 3 & \text{if } 30 < w \leq 50 \end{cases}$$

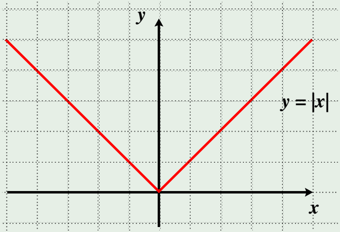
The range of the function C is obviously $\{2, 3\}$. See if you could sketch the graph of this *step function*, which is an example of *piece-wise constant functions*.

Some basic examples of functions

Example

The *absolute value function*, denoted by $f(x) = |x|$, is given by the following case by case formula:

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



$f(x) = |x|$ is an example of what people call *piecewise linear functions*. The domain of f is the set of all real numbers, and the range is the set of all non-negative real numbers.

Some basic examples of functions

Exercise

1. Sketch the graph of $f(x) = |2x - 4|$, which is another example of a piece-wise linear function.
2. Sketch the graph of the function ("*unit step function*")

$$y = u_c(t) = \begin{cases} 0 & \text{if } t < c \\ 1 & \text{if } t \geq c \end{cases}$$

   where c is some fixed constant.
3. Sketch the graph of the function $y = 3u_2(t) - 2u_4(t)$; and then use piece-wise defined formula to describe the function.

Example (real wold)

Let's just take the domain D to be an appropriate set of real numbers in each context below, and codomain $E = \mathbb{R}$ the set of all real numbers. All units are SI units when applicable.

numbers in D represent	function values in E represent
side length	area of the square with given side length
temperature in Celsius	same temperature in Fahrenheit

These functions above can be described more clearly by mathematical formulas if suitable symbols are introduced to denote the quantities involved.

numbers in D represent	function values in E represent
$x =$ side length	$A =$ area of the square...
$C =$ temperature in Celsius	$F =$ same temperature in Fahrenheit

$$A = x^2 \text{ (area of a square)}, \quad F = \frac{9}{5}C + 32 \text{ (unit conversion)}$$

Some basic examples of functions

Remark

Some domains of above functions are restricted to a certain range of values limited by physical restrictions.

1. *For the area function, we have* $D = \{x : x \geq 0\}$.
2. *The minimum temperature is taken as -273.15 on the Celsius scale.*

*It is important to understand the "*practical domain*" of a function in any modeling application, i.e., possible inputs of the function limited by the assumptions of the model, instead of just the "*natural domain*" of the function, i.e., where the formula makes sense mathematically.*

Outline

Some Elementary Functions

Following elementary mathematical functions you need to get familiar

- constant functions; e.g., 2, π, e.
- polynomial functions; e.g., $f(x) = x^3 + 2x^2 - 4x + 5$.
- rational functions; e.g., $f(x) = \dfrac{x^3 + 2x^2 - 4x + 5}{x^2 + 2x + 7}$.
- power functions; e.g., $f(x) = x^{3/2}$.
- exponential functions; e.g., $f(x) = 10^x$.
- logarithmic functions; e.g., $f(x) = \log_{10} x$.
- trigonometric functions; e.g., $\sin x$, $\cos x$, $\tan x$.
- inverse trigonometric functions; e.g., $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$.

Outline

Basic Operations: Sum, Product and Quotient

Given real-valued functions f and g, we can define new functions $f+g$ (*sum*), fg (*product*), and $\frac{f}{g}$ (*quotient*) simply by setting following rules:

$$(f+g)(x) = f(x) + g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

as long as both function values, $f(x)$ and $g(x)$, are well-defined, and the corresponding arithmetic operations on them are valid.

However, we need to be careful with the domains of these functions.

Domains of sum, product and quotient

1. For either $(f+g)(x)$ or $(fg)(x)$, the input value x must be in both the domain of f and the domain of g in order to have well-defined function values to add or to multiply. Hence the domain of $f+g$, or fg, is

$$\{x : x \text{ is is the domain of } f \text{ and } x \text{ is also in the domain of } g\}$$

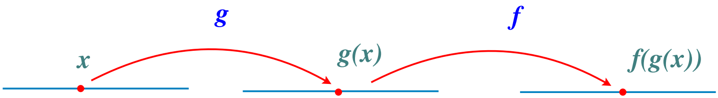
2. For $\dfrac{f(x)}{g(x)}$ to be well-defined, $f(x)$ and $g(x)$ have to be well-defined, and $g(x)$ has to be non-zero. Hence the domain of the function $\dfrac{f}{g}$ is

$$\{x : x \text{ is in the domain of } f, \text{ and } x \text{ is in the domain of } g, \text{ and } g(x) \neq 0\}$$

Basic Operations: Composition

One can also connect two "input-output machines" (functions) to form a new function, called the *composition* of f and g and denoted by the notation $f \circ g$, which is defined by

$$(f \circ g)(x) = f(g(x))$$



Obviously, we need $g(x)$ to be well-defined first, and then $g(x)$ to be in the domain of f in order to have a well-defined function value $f(g(x))$. Hence the domain of $f \circ g$ is given by

$$\begin{aligned}&\text{domain of } f \circ g\\ =&\{x : x \text{ is in the domain of } g \text{ and } g(x) \text{ is in the domain of } f\}\end{aligned}$$

Basic Operations: Composition

Some basic functions can simply be built by applying basic operations we mentioned to the constant functions and the linear function $f(x) = x$:

1. a constant function: $f(x) = 8,\ D = (-\infty, \infty)$
2. a linear function: $f(x) = 2x + 3,\ D = (-\infty, \infty)$
3. a quadratic function: $f(x) = 2x^2 - 4x + 8\ \ D = (-\infty, \infty)$
4. a polynomial function of degree 5: $g(x) = 3x^5 + 5x^4,\ D = (-\infty, \infty)$.
5. a rational function: $r(x) = \dfrac{x^2 + 4x + 4}{x^2 - 1}$, we require $x \neq \pm 1$ to avoid a zero denominator which leads to $D = (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.

Polynomial Function and Rational Function

1. A *polynomial function of degree n* is a function of the form

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

   where, and $a_0, \ldots, a_n$ are some constants.

2. A *rational function* is the quotient of two polynomials, i.e., a function of the form

$$R(x) = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}$$

   where n, m are non-negative integers, and $a_0, \ldots, a_n, b_0, \ldots, b_n$ are some constants with $a_n \neq 0$ and $b_m \neq 0$.

More Examples on Basic Operations of Functions

Example

Consider $f(x) = 2x - 1$, $g(x) = x^2 - 1$, then

$$(f+g)(x) = (2x-1) + (x^2-1) = 2x - x^2 - 2$$

$$(fg)(x) = (2x-1)(x^2-1) = 2x^3 - x^2 - 2x + 1$$

$$(5f)(x) = 5(2x-1) = 10x - 5$$

$$\left(\frac{f}{g}\right)(x) = \frac{2x-1}{x^2-1}$$

What are the domains of these functions?

Examples on Composition Operation

Example

Suppose $f(x) = 2x - 1$, $g(x) = x^2 - 1$ as above. Then

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(x^2 - 1) \\ &= 2(x^2 - 1) - 1 \\ &= 2x^2 - 3\end{aligned}$$

(Note that $f(\bigstar) = 2\bigstar - 1$)

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2x - 1) \\ &= (2x - 1)^2 - 1 \\ &= 4x^2 - 4x\end{aligned}$$

(Note that $g(\bigstar) = \bigstar^2 - 1$)

Note that in general, $f \circ g$ and $g \circ f$ are not the same function.

Examples on Composition Operation

Example

Let $f(x) = \dfrac{1}{x}$. Find $f \circ f$.

We have

$$(f \circ f)(x) = f(f(x)) = f\left(\frac{1}{x}\right) = \frac{1}{\frac{1}{x}} \stackrel{?}{=} x$$

Is the function $f \circ f$ the same as the function $h(x) = x$?

No! The domain of h is the set of all real numbers, but $x = 0$ is not in the domain of $f \circ f$, since $1/0$ is not a well-defined number.

If the answer to the function $f \circ f$ above is given in the "simplified form" of

$$(f \circ f)(x) = x,$$

it should be stated that there is actually a domain restriction $x \neq 0$.

Examples on Composition Operation

Example

Let $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{x+1}{x-2}$. Find $g \circ f$.

We have

$$(g \circ f)(x) = g(f(x)) = g\left(\frac{1}{x}\right) = \frac{\frac{1}{x}+1}{\frac{1}{x}-2}$$

with domain determined by the condition

$$x \neq 0 \quad \text{and} \quad \frac{1}{x} - 2 \neq 0$$

i.e., $x \neq 0, \dfrac{1}{2}$.

Example

Let $f(x) = x^2 - 3$, $g(x) = \sqrt{x-1}$. Find (i) $f \circ g$ and (ii) $g \circ f$.

Note that the domain of f is $(-\infty, \infty)$, and the domain of g is $[1, \infty)$.

(i) $(f \circ g)(x) = f(g(x)) = f(\sqrt{x-1}) = (\sqrt{x-1})^2 - 3$, with domain given by $x \geq 1$. Similarly, $(f \circ g)(x)$ is not exactly the same as the function $h(x) = x - 4$ since their domains are different.

(ii) $(g \circ f)(x) = g(f(x)) = g(x^2 - 3) = \sqrt{(x^2-3)-1}$. By working with the sign line of $(x^2 - 3) - 1 = (x-2)(x+2)$, the domain of $g \circ f$ can be found as $(-\infty, -2] \cup [2, \infty)$ from the condition that $(x-2)(x+2) \geq 0$.

What about the domain of $\sqrt[n]{b}$ (n-th root of b)?

More discussion on $\sqrt[n]{b}$

1. For any positive even number n, the *radical expression* $\sqrt[n]{b}$ denotes the positive root of the equation $x^n = b$. For example, $\sqrt[4]{16} = 2$ since $2^4 = 16$. No *real* root exists if b is negative, e.g., $\sqrt[4]{-16}$ does not exist as a real number since $x^4 \geq 0 > -16$ for any real number x, i.e., $x^4 = -16$ has no real solution.
2. For any positive odd number n, the equation $x^n = b$ has a unique *real* root for any given real number b, which is also denoted by $\sqrt[n]{b}$. For examples, $\sqrt[3]{8} = 2$ since $2^3 = 8$, and $\sqrt[3]{-8} = -2$ since $(-2)^3 = -8$.
3. Recall that a radical expression can also be expressed in terms of *exponent notation*; e.g., $\sqrt[n]{x} = x^{\frac{1}{n}}$ for any positive integer n. The relation between the *power function* $y = x^n$ and the *n-th root function* $y = \sqrt[n]{x}$ will be discussed in more detail later when we deal with the concept of *inverse function*.

More Discussion on Composition Operation

The composition of more than two functions can also be defined accordingly. For example, the composition $f \circ g \circ h$ is defined by

$$(f \circ g \circ h)(x) = f(g(h(x))) \ .$$

It is easy to see also that $f \circ g \circ h = f \circ (g \circ h) = (f \circ g) \circ h$. *What is the domain of this function*?

Exercise

Consider $f(x) = 2x - 1$, $g(x) = \dfrac{x^2+1}{x}$. Find the following functions, and determine their domains.

(a) $f(g(x))$ (b) $g(f(x))$ (c) $(f \circ g \circ f)(x) = f(g(f(x)))$

Outline

Even and Odd Functions

A function $y = f(x)$ is called an $\begin{cases} \text{even function if } f(-x) = f(x) \\ \text{odd function if } f(-x) = -f(x) \end{cases}$.

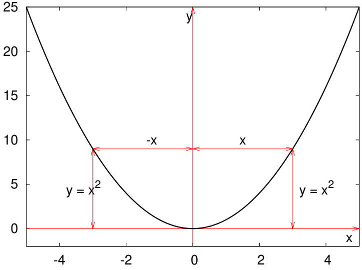
for all x in the domain of f.

Example

1. $y = x^2$ is an even function since $f(-x) = (-x)^2 = x^2 = f(x)$
2. $y = x^3$ is an odd function since $f(-x) = (-x)^3 = -x^3 = f(x)$
3. $y = |x|$ is an even function since $f(-x) = |-x| = |x| = f(x)$
4. $y = \dfrac{1}{x}$ is an odd function since $f(-x) = \dfrac{1}{-x} = -f(x)$ for $x \neq 0$

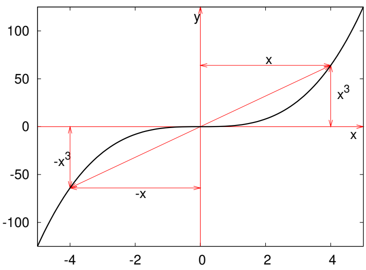
Even and Odd Functions

$(1) y = x^2$ is an even function



the graph of an even function is symmetric with respect to the y-axis (graph remains unchanged after reflection about y-axis)

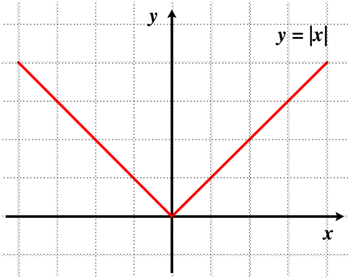
$(2) y = x^3$ is an odd function



the graph of an odd function is symmetric with respect to the origin (graph remains unchanged after rotation of 180 degrees about origin)

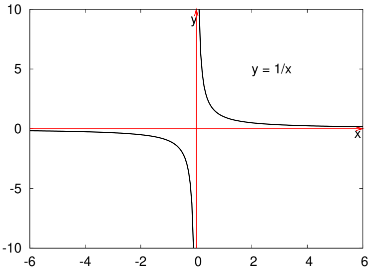
Even and Odd Functions

(1) $y = |x|$ is an even function



the graph of an even function is symmetric with respect to the y-axis

(2) $y = \frac{1}{x}$ is an odd function



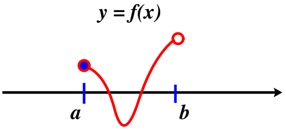
the graph of an odd function is symmetric with respect to the origin

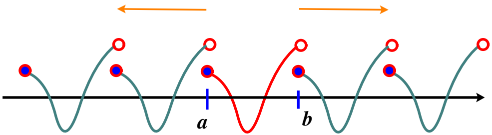
A function $f(x)$ is *periodic* if there is a number $T \neq 0$ such that $f(x+T) = f(x)$ for all x in the domain. The smallest such $T > 0$, if it exists, is called the *(fundamental) period* of the periodic function.

Note that a periodic function may has no (fundamental) period, why?

The graph of a periodic function does not change, if it is shifted to the left (or right), by a distance equal to an integral multiple of the period.

Any function f defined on the interval $[a, b)$ can be extended to a periodic function defined on the entire real line: keep shifting the graph by a distance of $b - a$.

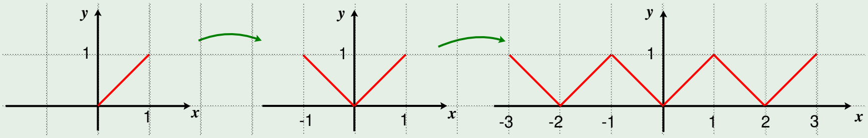




Examples of Periodic Functions

Examples

Given a function $f(x) = x$ defined for $0 \leq x \leq 1$. Extend $f(x)$ to the whole real line as an even periodic function of period 2.



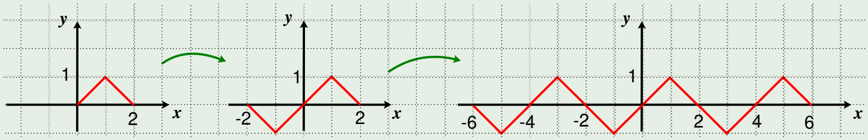
Examples of Periodic Functions

Example

Given a function

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ -x+2 & \text{if } 1 \le x \le 2 \end{cases}$$

defined on the interval $0 \le x \le 2$. Extend $f(x)$ to the whole real line as an odd periodic function of period 4.



Increasing and Decreasing Functions

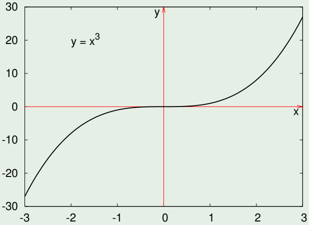
A function $y = f(x)$ is called

$$\begin{cases} \text{an } \textcolor{red}{\text{increasing function}} & \text{if } f(x_1) < f(x_2) \text{ whenever } x_1 < x_2 \\ \text{a } \textcolor{red}{\text{decreasing function}} & \text{if } f(x_1) > f(x_2) \text{ whenever } x_1 < x_2 \end{cases}$$

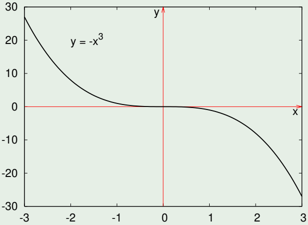
for all x_1, x_2 in the domain of f.

Example

increasing function: $y = x^3$



decreasing function: $y = -x^3$



The graph is rising/dropping when travelling along the positive x-direction. By arithmetic, $x_2^3 > x_1^3$ whenever $x_2 > x_1$, hence x^3 is an increasing function.

Outline

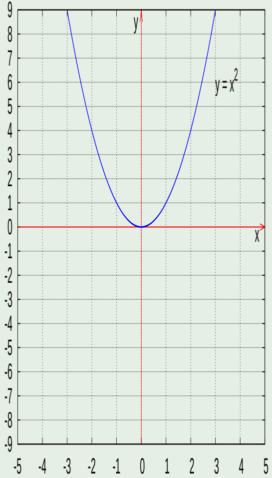
Transformations of Graphs

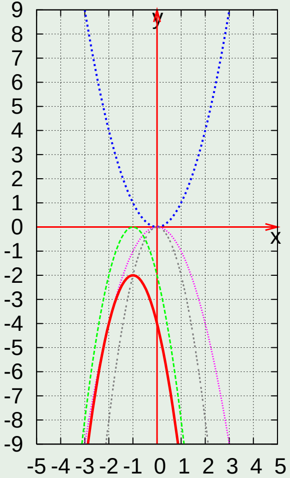
1. Graph of $y = f(x) + k$:
   $$\begin{cases} \text{upward shifting of the graph of } f \text{ by } k \text{ units if } k > 0 \\ \text{downward shifting of the graph of } f \text{ by } k \text{ units if } k < 0 \end{cases}$$
2. Graph of $y = f(x + k)$:
   $$\begin{cases} \text{shifting the graph of } f \text{ to the right by } |k| > 0 \text{ units if } k < 0 \\ \text{shifting the graph of } f \text{ to the left by } k \text{ units if } k > 0 \end{cases}$$
3. Graph of $y = -f(x)$: reflecting the graph of f across the x-axis.
4. Graph of $y = f(-x)$: reflecting the graph of f across the y-axis.
5. Graph of $y = kf(x)$, where $k > 0$:
   $$\begin{cases} \text{stretching the graph of } f \text{ in } y\text{-direction by a factor of } k \text{ if } k > 1 \\ \text{compressing the graph of } f \text{ in } y\text{-direction by a factor of } k \text{ if } 0 < k < 1 \end{cases}$$
6. Graph of $y = f(kx)$, where $k > 0$:
   $$\begin{cases} \text{compressing the graph of } f \text{ in } x\text{-direction by a factor of } k \text{ if } k > 1 \\ \text{stretching the graph of } f \text{ in } x\text{-direction by a factor of } k \text{ if } 0 < k < 1 \end{cases}$$

Example (completing the square)

Given the graph of $y = x^2$, sketch the graph of $y = -2(x+1)^2 - 2$ by using suitable transformations of graphs. Consider the following sequence of transformations:

$$\boxed{y = x^2} \downarrow \boxed{y = -x^2} \downarrow \boxed{y = -2x^2} \downarrow \boxed{y = -2(x+1)^2} \downarrow \boxed{y = -2(x+1)^2 - 2}$$





Example (completing the square, continued)

A quadratic polynomial $y = ax^2 + bx + c$, where $a \neq 0$, can be written as

$$y = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

Thus the vertex of its graph is given by the coordinate point

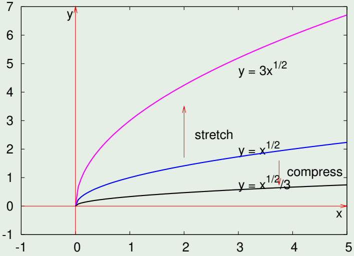
$$\left(-\frac{b}{2a}, c - \frac{b^2}{4a}\right),$$

which is the lowest point on the graph if $a > 0$, and highest point on the graph if $a < 0$. The graph is symmetric with respect to the vertical line $x = -\frac{b}{2a}$, the *axis of symmetry*.

Example

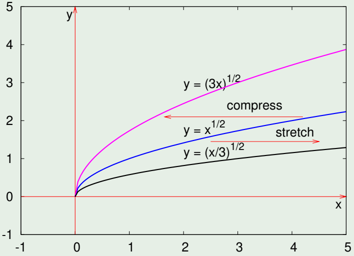
Consider the function defined by $f(x) = \sqrt{x}$. Compare the graph of $y = f(x)$ with the graphs of $y = 3f(x)$,$y = \frac{1}{3}f(x)$, $y = f(3x)$ and $y = f\left(\frac{x}{3}\right)$:

$3f(x) \longleftrightarrow f(x) \longleftrightarrow \frac{1}{3}f(x)$



$y = 3\sqrt{x}, y = \sqrt{x}, y = \frac{1}{3}\sqrt{x}$

$f(3x) \longleftrightarrow f(x) \longleftrightarrow f\left(\frac{x}{3}\right)$



$y = \sqrt{3x}, y = \sqrt{x}, y = \sqrt{\frac{x}{3}}$

Calculus IB: Lecture 03

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Outline

Outline

One-to-One Functions

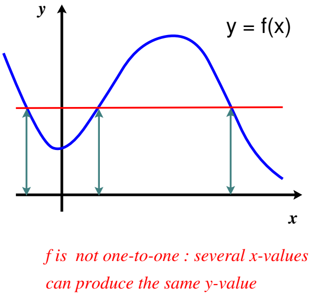
1. Recall that for any x in the domain of a function f, only one function value $f(x)$ can be assigned to x.
2. However, it is possible that two different numbers $x_1 \neq x_2$ in the domain of f have the same function value, i.e. $f(x_1) = f(x_2)$.
3. By ruling out above possibility, we have the concept of a *one-to-one function*: A function f is said to be *one-to-one* if $f(x_1) \neq f(x_2)$ for *any* two numbers $x_1 \neq x_2$ in the domain of f.
4. In other words, $f(x)$ never takes on the same function value twice or more times when x runs through the domain of f; or equivalently, the equation

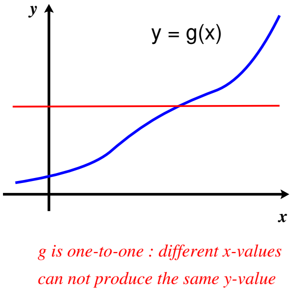
$$f(x) = b$$

has exactly one solution for any b in the range of f. In particular, f is a one-to-one function if $x_1 = x_2$ whenever $f(x_1) = f(x_2)$.

One-to-One Functions

Graphically speaking, we have the *Horizontal Line Test* which says that f is a one-to-one function if every horizontal line hits the graph of f at most once.





Examples of One-to-One Functions

Example

Let f be defined by $f(x) = x^2$. It is obviously that f takes on every positive number $b \neq 0$ exactly two times, since $x^2 = b$ has exactly two roots $x = \pm\sqrt{b}$ for any $b > 0$; e.g.,

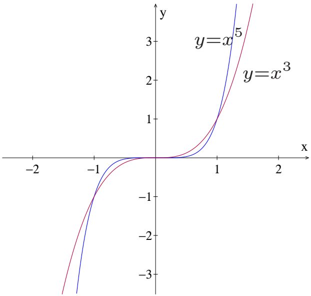
$$f(2) = 2^2 = 4 = (-2)^2 = f(-2) \ .$$

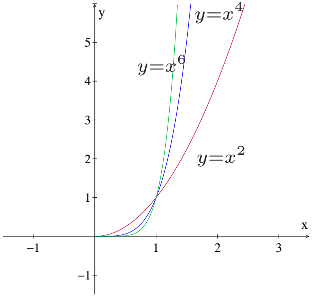
f is thus not a one-to-one function.

However, if the domain of $f(x) = x^2$ is *restricted* to $0 \leq x < \infty$, the function f is then one-to-one, since $x^2 = b$ has exactly one non-negative solution $\sqrt{b}$.

Increasing (Decreasing) Functions are One-to-One

1. If f is an increasing function with domain D a set of real numbers, then $f(x_1) < f(x_2)$ for any number x_1, x_2 in the domain D such that $x_1 < x_2$. Hence $f(x_1) \neq f(x_2)$ for any $x_1 \neq x_2$.
2. Similarly, the decreasing functions are also one-to-one functions.
3. For any positive integer n, the *power function* $y = x^{2n+1}$ is an increasing function with domain $-\infty < x < \infty$. Similarly, the power function $y = x^{2n}$ is an increasing function when the domain is restricted to $0 \leq x < \infty$.





Outline

Power Functions

Note that for any positive integer n, the function $\dfrac{1}{x^n}$ can also be expressed in the form of power function as $\dfrac{1}{x^n} = x^{-n}$.

The *exponent laws* for integer powers (or exponents) then follow easily:

$$\text{(i)}\ \ x^0 = 1 \text{ (by convention)} \quad \text{(ii)}\ \ x^{n+m} = x^n x^m \quad \text{(iii)}\ \ x^{n-m} = \frac{x^n}{x^m}$$

$$\text{(iv)}\ \ (x^n)^m = x^{nm} \quad \text{(v)}\ \ (xy)^n = x^n y^n \quad \text{(vi)}\ \ \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

where n, m are any integers.

Power Functions

For example, if n, m are positive integers with $n < m$, then

$$x^n x^m = (\underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ many factors}}) \cdot (\underbrace{x \cdot x \cdot \cdots \cdot x}_{m \text{ many factors}}) = \underbrace{x \cdot x \cdot \cdots \cdot x}_{n+m \text{ many factors}} = x^{n+m}$$

$$\frac{x^n}{x^m} = \frac{\overbrace{x \cdot x \cdot \cdots \cdot x}^{n \text{ many factors}}}{\underbrace{x \cdot x \cdot \cdots \cdot x}_{m \text{ many factors}}} = \frac{1}{\underbrace{x \cdot x \cdot \cdots \cdot x}_{m-n \text{ many factors}}} = \frac{1}{x^{m-n}} = x^{n-m}$$

Note that these exponent laws hold also for exponents which are real numbers. However, it would be harder to see what x^p means when p is a irrational number (such as $p = \sqrt{2}$).

Outline

Inverse Functions Arising from One-to-One Functions

Consider the linear relation

$$y = 2x + 3$$

between x and y, where y is considered as a function of x, can be rewritten as

$$x = \frac{y-3}{2}$$

Hence x can then be considered as a function of y.

The same process can be applied to any one-to-one functions.

Inverse Functions Arising from One-to-One Functions

If f is a one-to-one function, then for any b in the range of f, the equation $f(x) = b$ has exactly one solution in the domain of f.

We can therefore define *inverse function* of f, usually denoted by f^{-1} (Warning: the symbol f^{-1} here does not mean $\frac{1}{f}$), by reversing the roles of the domain and range of f as follows:

$$f^{-1}: \begin{array}{c} \text{range of } f \\ \| \\ \text{domain of } f^{-1} \end{array} \longrightarrow \begin{array}{c} \text{domain of } f \\ \| \\ \text{range of } f^{-1} \end{array}$$

where

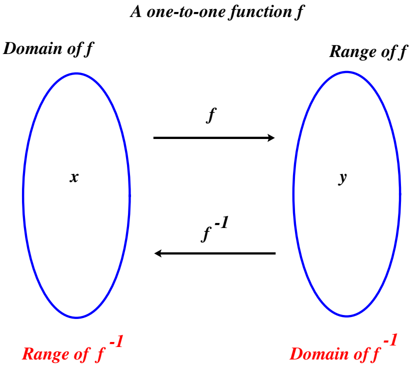
$$f^{-1}(b) = \text{the unique solution of the equation } f(x) = b$$

for any b in the domain of f^{-1} (i.e., the range of f).

Inverse Functions and Arrow Diagram

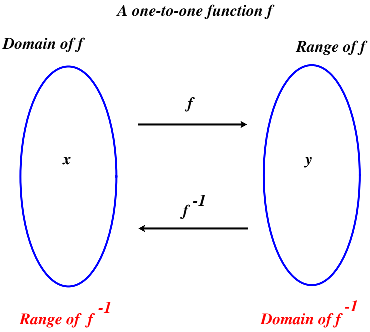
Suppose we use an arrow diagram to represent a function f, which assigns to any given number x in the domain of f a unique number y in the range of f (i.e., $y = f(x)$). Then, defining f^{-1} is just like "reversing the arrow" of f:

- turning the range of f into the domain of the inverse function f^{-1};
- turning the domain of f into the range of the inverse function f^{-1};
- $x = f^{-1}(y)$ coming from the unique solution of $f(x) = y$



Inverse Functions and Arrow Diagram

1. A one-to-one function $y = f(x)$ gives rise to a one-to-one matching of the numbers in two sets.
2. Depending on which variable you take as independent variable, you have either the original function $f(x)$, or the inverse function $f^{-1}(y)$.
3. The following properties of f and f^{-1} follow easily from chasing the arrows: $f^{-1}(f(x)) = x$ for any x in the domain of f; and $f(f^{-1}(y)) = y$ for any y in the range of f.



Examples of Inverse Functions

Example

Find the inverse function $f^{-1}(x)$ for the function $f(x) = \dfrac{3x+2}{2x-1}$. Let $y = \dfrac{3x+2}{2x-1}$, then $y(2x-1) = 3x+2 \Longleftrightarrow (2y-3)x = y+2$.

Hence we have

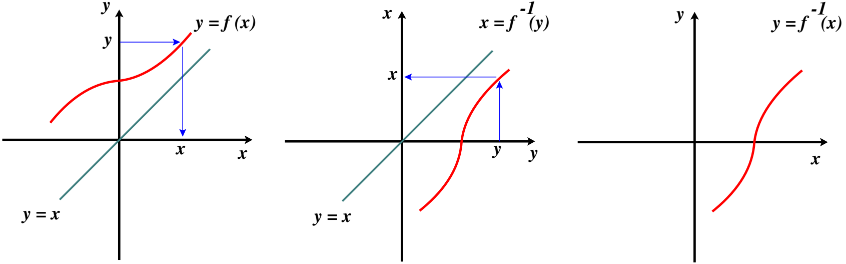
$$x = \frac{y+2}{2y-3} = f^{-1}(y) \quad \textit{and} \quad f^{-1}(x) = \frac{x+2}{2x-3}.$$

The domain of f^{-1}, which is the range of f, is given by $x \neq \dfrac{3}{2}$; i.e., $\left(-\infty, \dfrac{3}{2}\right) \bigcup \left(\dfrac{3}{2}, \infty\right)$. And the range of f^{-1}, which is the domain of f, is given by $x \neq \dfrac{1}{2}$; i.e., $\left(-\infty, \dfrac{1}{2}\right) \bigcup \left(\dfrac{1}{2}, \infty\right)$

Graphs of Inverse Functions

It is interesting that the graph of $x = f^{-1}(y)$ is the same as the graph of $y = f(x)$, except that the y-axis is now viewed as the domain axis.

In particular, the graph of the inverse function $y = f^{-1}(x)$ can be obtained by reflecting the graph of the one-to-one function $y = f(x)$ across the line $y = x$, or simply by renaming the x-axis as the y-axis, and y-axis as the x-axis.



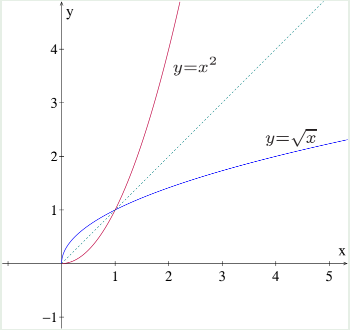
Reflecting range into domain *Renaming the axes*

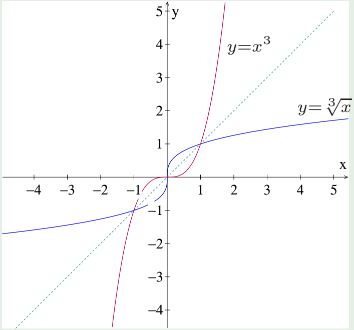
Graphs of Inverse Functions

Example

For any integer $n \geq 2$, the *n-th root function* $\sqrt[n]{x}$ is defined as follows:

$$\sqrt[n]{x} = \begin{cases} \text{the inverse function of } y = x^n \text{ with domain } [0, \infty) & \text{if } n \text{ is even} \\ \text{the inverse function of } y = x^n & \text{if } n \text{ is odd} \end{cases}$$





In particular, the domain of an n-th root function is given as follows.

$$\text{domain of } \sqrt[n]{x} \text{ is given by: } \begin{cases} [0, \infty) & \text{if } n \text{ is even} \\ (-\infty, \infty) & \text{if } n \text{ is odd} \end{cases}$$

Using exponent notation, an n-th root function can be written as

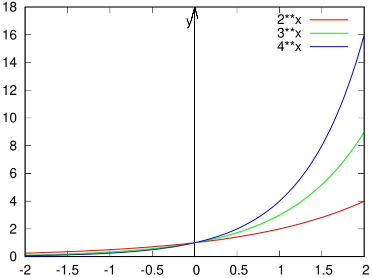
$$\sqrt[n]{x} = x^{\frac{1}{n}}.$$

More generally, a *power function* of the form $x^{\frac{n}{m}}$, where n is an integer and m is a positive integer, is defined by

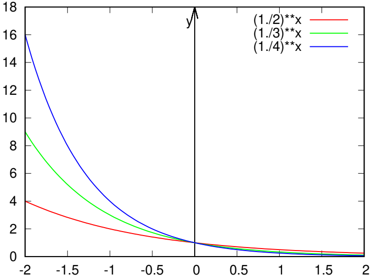
$$x^{\frac{n}{m}} = \sqrt[m]{x^n} \ .$$

Outline

For any positive real number $a \neq 1$, the *exponential function with base a* is given by $y = a^x$, whose graphs are shown as follows.



$y = 2^x,\ y = 3^x,\ y = 4^x$
$(a > 1)$



$y = (\frac{1}{2})^x,\ y = (\frac{1}{3})^x,\ y = (\frac{1}{4})^x$
$(0 < a < 1)$

1. The domain of $y = a^x$ is $(-\infty, \infty)$.
2. The range of $y = a^x$ is $(0, \infty)$.
3. We also have

$$y = a^x = \begin{cases} \text{is an increasing function} & \text{if } a > 1, \\ \text{is a decreasing function} & \text{if } 0 < a < 1. \end{cases}$$

4. Since many expressions with negative a like $(-1)^{1/2}$ it not a real number, and since $a = 0$ leads to a trivial constant function, we usually only consider the case of $a > 0$.

An exponential function $y = a^x$ must be one-to-one (try to prove it), and hence has an inverse function, which is denoted by $x = \log_a y$, by reversing the roles of the domain and range:

$$\begin{cases} y = a^x \\ \text{domain: } -\infty < x < \infty \\ \text{range: } y > 0 \end{cases}$$

$$\longleftrightarrow \begin{cases} x = \log_a y \\ \text{domain: } y > 0 \\ \text{range: } -\infty < x < \infty \end{cases}$$

$$\longleftrightarrow \begin{cases} y = \log_a x \\ \text{domain: } x > 0 \\ \text{range: } -\infty < y < \infty \end{cases}$$

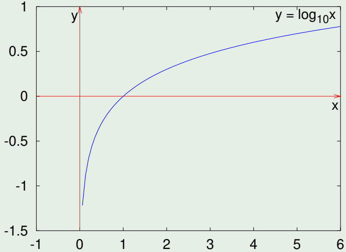
Example (take $a = 10$ and see what happens)

$y = 10^x \downarrow$

x	-3	-2	-1	0	$c = ?$	1	2	$b = ?$
y	0.001	0.01	0.1	1	8	10	100	100

$\uparrow x = \log_{10}(y)$

The graph of the *exponential function with base 10*, $y = 10^x$, gives you the graphs of the *common logarithmic function* $y = \log_{10} x$ at the same time.



Note that to find the value of $b = \log_{10} 100$ is just a problem of solving the equation $10^b = 100 = 10^2$, and hence obviously $b = 2 = \log_{10} 100$.
It is not so easy to find the exact value of $c = \log_{10} 8$ though, which means $10^c = 8$. A rough estimate is $\frac{1}{2} < c < 1$ since $10^{1/2} < 8 = 10^c < 10^1$.

Properties of Exponential and Logarithmic Functions

Once you understand how to convert exponential relationship into logarithmic relationship, and vice versa,

$$y = a^x \quad \longleftrightarrow \quad x = \log_a y$$

the following properties of logarithms are easy to verify.

Exponential Function	Logarithmic Function
$a^0 = 1$	$\log_a 1 = 0$
$a^1 = a$	$\log_a a = 1$
$a^x = a^x$	$\log_a a^x = x$
$a^{\log_a x} = x$	$\log_a x = \log_a x$
$a^x a^y = a^{x+y}$	$\log_a xy = \log_a x + \log_a y$
$\frac{a^x}{a^y} = a^{x-y}$	$\log_a \frac{x}{y} = \log_a x - \log_a y$
$(a^x)^y = a^{xy}$	$\log_a x^y = y \log_a x$
	$\log_c x = \frac{\log_a x}{\log_a c}$

Verify the property $\log_a(xy) = \log_a x + \log_a y$

Let $C = \log_a x$, and $D = \log_a y$. Hence we have $a^C = x$ and $a^D = y$. What if you multiplying the two together?

$$a^C a^D = xy \iff a^{C+D} = xy$$

Now, convert it to:

$$\log_a(xy) = C + D = \log_a x + \log_a y$$

All other properties in the table above can be checked by similar arguments. (Exercise!)

The Natural Exponential/Logarithmic Function

The exponential/logarithmic function with a special base $e \approx 2.7182\ldots$

$$y = e^x, \qquad y = \log_e x \triangleq \ln x$$

is called the *natural exponential/logarithmic function*. Note that all other exponential function can be expressed in term of the natural exponential function, since

$$a^x = e^{\ln a^x} = e^{x \ln a}\ .$$

For example, $3^x = e^{x \ln 3}$.

The Natural Exponential/Logarithmic Function

Why are we interested in the very special number $e \approx 2.7182\ldots$?

More precisely, e can be defined as the "limit" as follows

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

We shall discuss the topic about "e" in more detail later.

More Examples on Using Exp-Log

Example

Find the domain and range of the function $y = f(x) = 2\ln(5-x)+1$. What is its inverse function?

Recall that $\log_a(\bigstar)$ is well-defined if and only if $\bigstar > 0$. Hence the domain of $f(x)$ is given by: $5 - x > 0$, i.e., $x < 5$. We also have

$$\begin{aligned} & y = 2\ln(5-x)+1 \\ \Longrightarrow & \frac{y-1}{2} = \ln(5-x) \\ \Longrightarrow & 5 - x = e^{\frac{y-1}{2}} \\ \Longrightarrow & x = 5 - e^{\frac{y-1}{2}} \end{aligned}$$

i.e., the inverse function $x = f^{-1}(y) = 5 - e^{\frac{y-1}{2}}$. The range of $f(x)$ is the domain of $f^{-1}(y)$, which is the set of all real numbers.

More Examples on Using Exp-Log

Example

Solve the following equations: (a) $24(1 - e^{-t/2}) = 16$; (b) $2^{2x-3} = 3^{x+1}$

$$24(1 - e^{-t/2}) = 16$$
$$1 - e^{-t/2} = \frac{16}{24} = \frac{2}{3}$$
$$e^{-t/2} = \frac{1}{3}$$
$$-\frac{t}{2} = \ln\frac{1}{3}$$
$$t = -2\ln\frac{1}{3} = \ln 9 \quad (\approx 2.1792)$$

$$2^{2x-3} = 3^{x+1}$$
$$\ln 2^{2x-3} = \ln 3^{x+1}$$
$$(2x-3)\ln 2 = (x+1)\ln 3$$
$$(2\ln 2 - \ln 3)x = \ln 3 + 3\ln 2$$
$$x = \frac{\ln 3 + 3\ln 2}{2\ln 2 - \ln 3}$$

Hyperbolic Functions

The hyperbolic functions are defined and denoted by

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

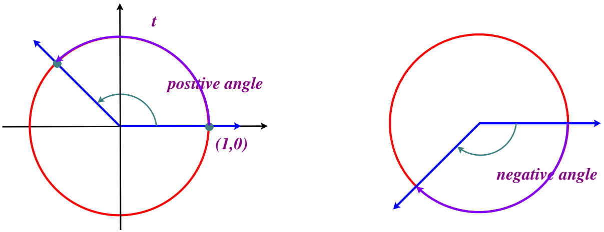
Please verify the following identities:

- $\cosh^2 x - \sinh^2 x = 1$
- $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
- $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

Hyperbolic functions have some similar properties to *trigonometric functions*.

Outline

If the point $(1, 0)$ starts to travel along the unit circle centered at the $(0, 0)$ through a distance θ in counterclockwise direction, the angle subtended by the corresponding circular arc is said to be a positive angle with *radian measure* θ.
Angles obtained by clockwise rotations are considered as negative angles.



Directed angle : angle can be assigned a $+$ve or -ve sign

Radian Measure of an Angle

Radian is a measure of an angle by circular arc length along the unit circle.

1. Recall that the length of a unit circle is 2π. Thus the radian measure of a 360° angle is 2π, and -2π if the angle is -360°.
2. In proportion, the degree measure and radian measure of an angle can be converted to each other according to

$$\frac{\text{radian measure}}{\text{degree measure}} = \frac{2\pi}{360} = \frac{\pi}{180}$$

3. In particular, we have

$$360^\circ = 2\pi \text{ rad} \qquad 180^\circ = \pi \text{ rad} \qquad 45^\circ = \frac{\pi}{4} \text{ rad}$$

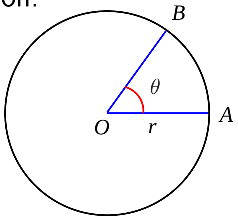
$$30^\circ = \frac{\pi}{6} \text{ rad} \qquad 60^\circ = \frac{\pi}{3} \text{ rad} \qquad -90^\circ = \frac{\pi}{2} \text{ rad}$$

Radian Measure of an Angle

Since the length and area of a circle of radius r are $2\pi r$ and πr^2, the arc length and area of a circular section subtended by an angle θ in radians can be determined according to the following proportion:

$$\frac{\text{circular sector area}}{\text{circle area}} = \frac{\theta}{2\pi} = \frac{\text{circular arc length}}{\text{circle length}}$$

$$\frac{\text{circular sector area}}{\pi r^2} = \frac{\theta}{2\pi} = \frac{\text{circular arc length}}{2\pi r}$$



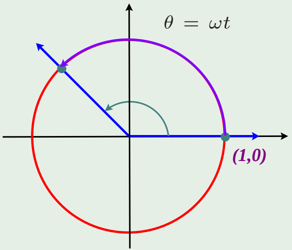
and hence we have

$$\text{circular sector area} = \frac{1}{2} r^2 \theta \quad \text{and} \quad \text{circular arc length} = r\theta$$

where θ is measured in radians, NOT degrees.

Example

If a particle is moving along a unit circle with *angular velocity* ω radians per second, then the angle subtended after t seconds is given by $\theta = \omega t$ radians, which is the distance traveled by the particle.



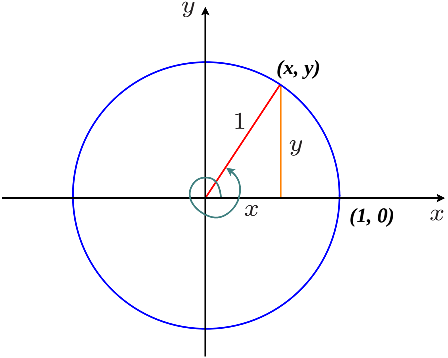
If the radius of the circle is R, then the distance traveled by the particle after t seconds is $R\omega t$.

Outline

Sine and Cosine Functions

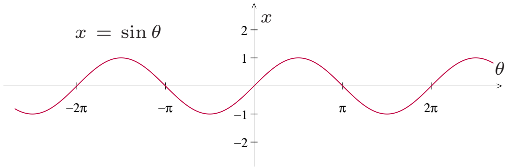
When a point originally at $(0, 1)$ moves along the unit circle through an angle of θ radians, the coordinates of the position (x, y) reached by the point depend on the value of θ, i.e., they are functions of θ:

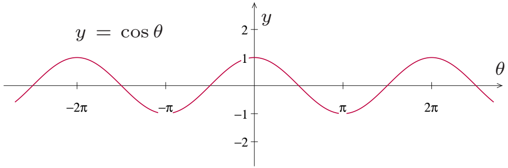
$$y = \sin\theta \quad \text{and} \quad x = \cos\theta, \quad \text{where } \theta \in (-\infty, +\infty) \text{ and } x, y \in [-1, 1].$$



Sine and Cosine Functions

It is easy to plot the graphs of $x = \sin\theta$ and $y = \cos\theta$ from the geometry of the circle.

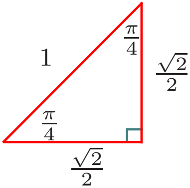


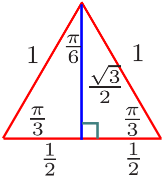


Since θ and $2\pi + \theta$ give you the same point on the unit circle, we have $\sin(\theta + 2\pi) = \sin\theta$ and $\cos(\theta + 2\pi) = \cos\theta$ i.e., both functions are periodic with period 2π.

Some Function Values of $\sin\theta$ and $\cos\theta$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	-1



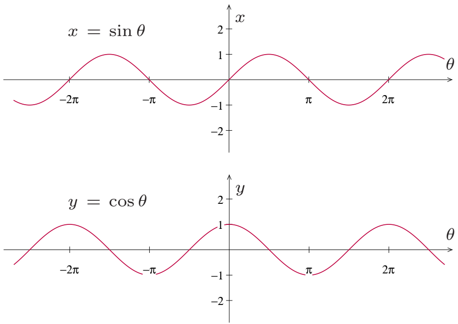


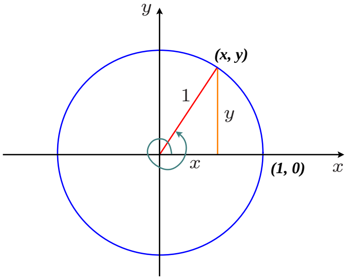
- $\sin\theta = 0$ if and only if $\theta = n\pi$ for some integer n. (points on the unit circle with zero y-coordinates are $(\pm 1, 0)$)
- $\cos\theta = 0$ if and only if $\theta = (2n+1)\frac{\pi}{2} = (n+\frac{1}{2})\pi$ for some integer n. (points on the unit circle with zero x-coordinates are $(0, \pm 1)$)

Properties of Sine and Cosine

Note that we have the following identities:

1. $\sin^2\theta + \cos^2\theta = 1$ (Pythagoras Theorem)
2. $\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right)$ (graph shifting)
3. $\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$ (graph shifting)

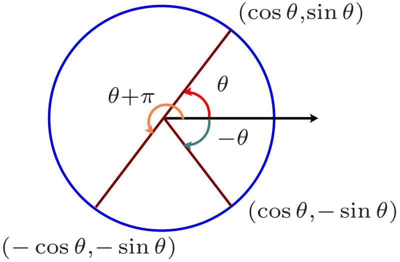




Properties of Sine and Cosine

Since θ and $-\theta$ put two points on unit circle symmetric with respect to x-axis and $\theta + \pi$ gives a point antipodal to that of θ, we have

$$\sin(-\theta) = -\sin\theta,$$
$$\cos(-\theta) = \cos\theta,$$
$$\sin(\theta+\pi) = -\sin\theta,$$
$$\cos(\theta+\pi) = -\cos\theta.$$

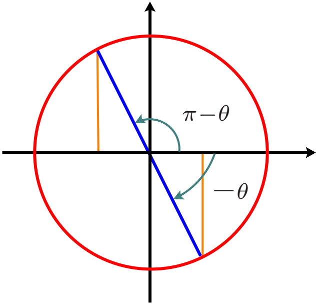


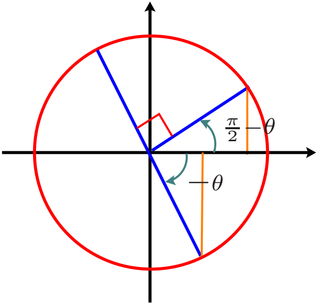
It is easy to see that $\sin\theta$ is an odd function and $\cos\theta$ is an even function.

By studying points on the unit circle given by the angles θ, $\frac{\pi}{2}-\theta$ and $\pi-\theta$, we have:

$$\sin(\pi-\theta)=\sin\theta, \qquad \sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta,$$

$$\cos(\pi-\theta)=-\cos\theta, \qquad \cos\left(\frac{\pi}{2}-\theta\right)=\sin\theta.$$

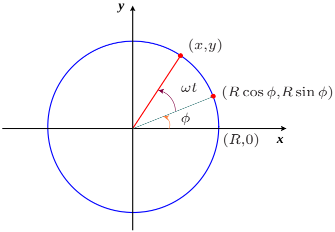




Properties of Sine and Cosine

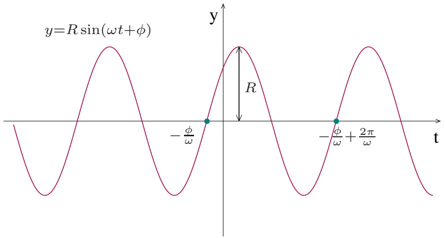
1. If (x, y) is a point on the circle of radius R, with the equation $x^2 + y^2 = R^2$, we have by proportion that
$$x = R\cos\theta, \qquad y = R\sin\theta\ .$$
2. If the point is rotating around the circle with constant angular velocity ω from $(R, 0)$, then at time t, the x and y coordinates of the point are given by
$$x = R\cos(\omega t), \qquad y = R\sin(\omega t)\ .$$
3. If the initial position of the point is $(R\cos\phi, R\sin\phi)$ instead of $(R, 0)$, the coordinate functions of the point are given by
$$x = R\cos(\omega t + \phi), \qquad y = R\sin(\omega t + \phi)\ .$$

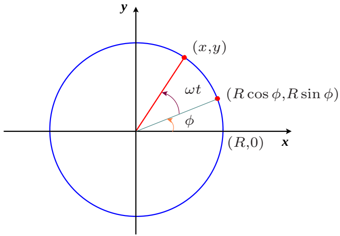
The rotation is
$$\begin{cases} \text{counterclockwise}, & \text{if } \omega > 0, \\ \text{clockwise}, & \text{if } \omega < 0. \end{cases}$$



Properties of Sine and Cosine

1. It is clear that functions $x = R\cos(\omega t + \phi)$ and $y = R\sin(\omega t + \phi)$ are periodic with period $2\pi/|\omega|$.
2. Such functions are often used in describing certain periodic oscillation motion, namely, *simple harmonic motion*. R is called the *amplitude*, $-\phi/\omega$ the *phase shift* and $\omega t + \phi$ the *phase* or *phase angle*.
3. The graphs of these functions can be easily found by performing suitable transformations on the graph of the sine or cosine function.





Calculus IB: Lecture 04

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Outline

Outline

More Trigonometric Functions

Four other trigonometric functions, namely, $\tan\theta$ (*tangent*), $\cot\theta$ (*cotangent*), $\csc\theta$ (*cosecant*), and $\sec\theta$ (*secant*) are defined by

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$	domain: $\{\theta : \cos\theta \neq 0\}$ range: $(-\infty, \infty)$
$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	domain: $\{\theta : \sin\theta \neq 0\}$ range: $(-\infty, \infty)$
$\csc\theta = \dfrac{1}{\sin\theta}$	domain: $\{\theta : \sin\theta \neq 0\}$ range: $(-\infty, -1] \cup [1, \infty)$
$\sec\theta = \dfrac{1}{\cos\theta}$	domain: $\{\theta : \cos\theta \neq 0\}$ range: $(-\infty, -1] \cup [1, \infty)$

Properties of $\tan\theta$

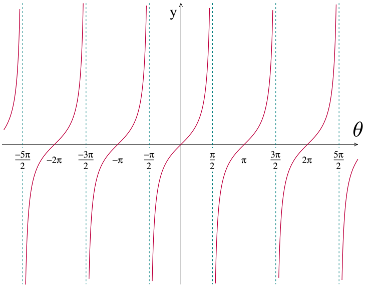
The function $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is a periodic function with period π:

$$\tan(\theta+\pi) = \tan\theta.$$

The domain of $\tan\theta$ is $\theta \neq n\pi + \pi/2$, where $n = 0, \pm1, \pm2, \pm3, \cdots$, and the range is $(-\infty, \infty)$.

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	$--$

$\tan(-\theta) = -\tan\theta$ (odd function)



Properties of $\tan\theta$, $\cot\theta$, $\sec\theta$ and $\csc\theta$

In addition to the identity $\sin^2\theta + \cos^2\theta = 1$, we have the identities:

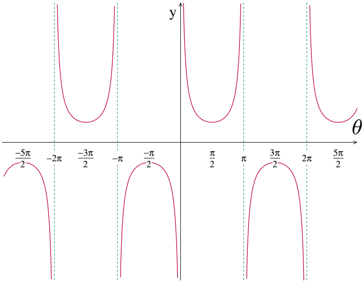
$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

For example, we can prove

$$1 + \tan^2\theta = 1 + \frac{\sin^2\theta}{\cos^2\theta} = \frac{\cos^2\theta + \sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} = \sec^2\theta.$$

You can prove the second one by yourself.

Graphs of $\csc\theta$ and $\sec\theta$



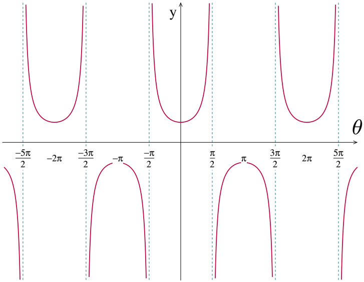
$$y = \csc\theta = \frac{1}{\sin\theta}$$

$$\text{period } = 2\pi$$

$$\text{domain } \theta \neq n\pi$$

$$\text{where } n = 0, \pm1, \pm2, \cdots$$

$$\text{range } x \leq -1 \text{ or } x \geq 1$$



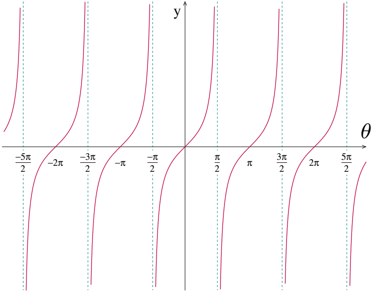
$$y = \sec\theta = \frac{1}{\cos\theta}$$

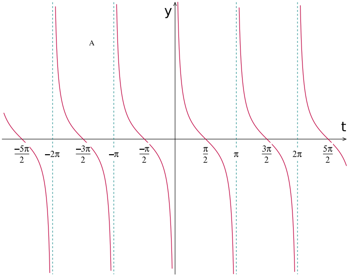
$$\text{period } = 2\pi$$

$$\text{domain } \theta \neq n\pi + \frac{\pi}{2}$$

$$\text{where } n = 0, \pm1, \pm2, \cdots$$

$$\text{range } x \leq -1 \text{ or } x \geq 1$$





$$y = \tan\theta = \frac{\sin\theta}{\cos\theta}$$

period $= \pi$

domain $\theta \neq n\pi + \frac{\pi}{2}$

where $n = 0, \pm1, \pm2, \cdots$

range $(-\infty, +\infty)$

$$y = \cot x = \frac{\cos\theta}{\sin\theta}$$

period $= \pi$

domain $\theta \neq n\pi$

where $n = 0, \pm1, \pm2, \cdots$

range $(-\infty, +\infty)$

Trigonometric Identities: Angle Addition and Subtraction

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$

$$\cos 2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

Trigonometric Identities: Product to Sum/Sum to Product

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha+\beta) - \cos(\alpha-\beta)\right]$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

Trigonometric Identities

All these formulas can be derived from one identity

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\beta\sin\alpha$$

For examples,

$$\begin{aligned}
&\cos(\alpha-\beta) = \cos(\alpha+(-\beta)) \\
=&\cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta) \\
=&\cos\alpha\cos\beta + \sin\beta\sin\alpha
\end{aligned}$$

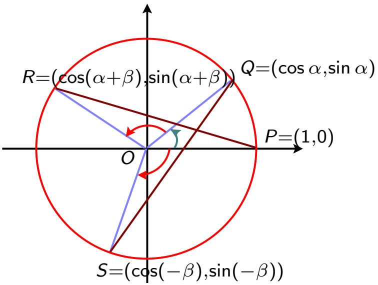
$$\begin{aligned}
&\sin(\alpha-\beta) \\
=&\cos\left(\left(\frac{\pi}{2}-\alpha\right)+\beta\right) \\
=&\cos\left(\frac{\pi}{2}-\alpha\right)\cos\beta - \sin\left(\frac{\pi}{2}-\alpha\right)\sin\beta \\
=&\sin\alpha\cos\beta - \sin\beta\cos\alpha
\end{aligned}$$

The Proof of $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\beta\sin\alpha$

Note that the two triangles $\triangle POR$ and $\triangle SOQ$ are congruent, as you can rotate one to the other by an angle of β. In particular, we have $PR = SQ$ and $PR^2 = SQ^2$.

Recall that the distance between two points (x_1, y_1) and (x_2, y_2) is given by

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



The identity follows then from $PR^2 = SQ^2$:

$$\begin{aligned}
(\cos(\alpha+\beta)-1)^2 + (\sin(\alpha+\beta)-0)^2 &= (\cos\alpha - \cos(-\beta))^2 + (\sin\alpha - \sin(-\beta))^2 \\
\cos^2(\alpha+\beta) - 2\cos(\alpha+\beta) + 1 + \sin^2(\alpha+\beta) &= \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta \\
&\quad + \sin^2\alpha + 2\sin\alpha\sin\beta + \sin^2\beta \\
2 - 2\cos(\alpha+\beta) &= 2 - 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta
\end{aligned}$$

Some Exercises

1. Work out the triple angle formulas for $\sin 3\alpha$, $\cos 3\alpha$.
   Hint: $\sin 3\alpha = \sin(\alpha + 2\alpha)$
2. Can you rewrite functions like $y = a\sin\omega t + b\cos\omega t$ into the form $y = R\sin(\omega t + C)$ for some constants R, ω, C? For example, since

   $\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, we have

   $$y = \sin t + \cos t = \sqrt{2}(\sin t\cos\frac{\pi}{4} + \cos t\sin\frac{\pi}{4}) = \sqrt{2}\sin(t + \frac{\pi}{4}) .$$

   Hint: consider

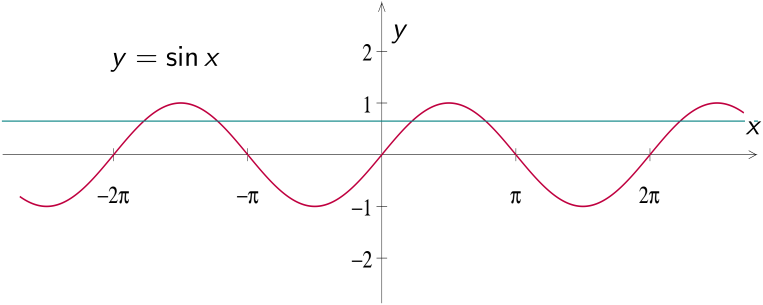
   $$a\sin\omega t + b\cos\omega t = \sqrt{a^2+b^2}\left[\frac{a}{\sqrt{a^2+b^2}}\sin\omega t + \frac{b}{\sqrt{a^2+b^2}}\cos\omega t\right]$$

   which can be rewritten as $R\sin(\omega t + C)$, or $R\cos(\omega t + C)$ for suitable choice of C.

Outline

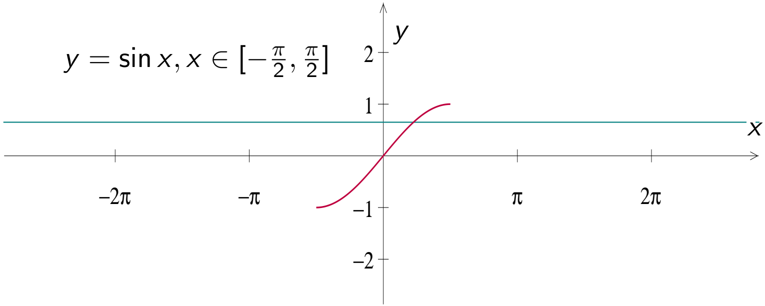
Inverse Trigonometric Functions

The horizontal line test shows that $\sin x$, $\cos x$, or $\tan x$ have no inverse function in general. For example, the periodic function $\sin x$ is obviously not one-to-one.



Inverse Trigonometric Functions

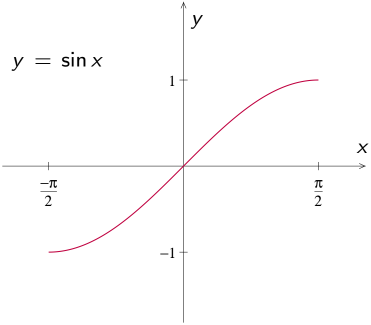
However, after restricting the domain to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, the function $y = \sin x$ is one-to-one, and hence has an inverse function.

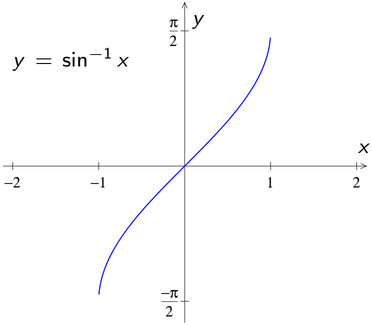


The inverse function of $y = \sin x$, with x restricted to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$, is denoted by $y = \sin^{-1} x$ or $y = \arcsin x$.

Graphs of $\sin\theta$ and $\sin^{-1}\theta$

Recall that the graph of $y = \sin^{-1} x$ can be found by reflecting the part of the graph of $y = \sin x$, with $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, across the line $y = x$.

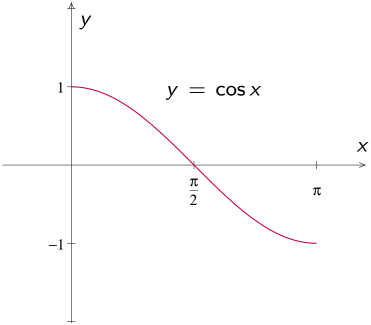


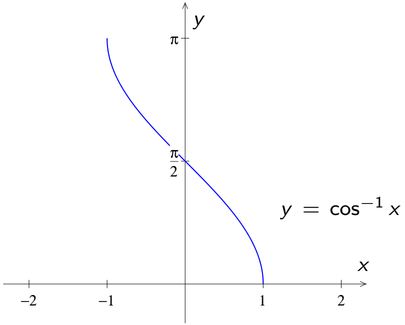


We can shift the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in theoretical, but we always use $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ in definition by convention.

Graphs of $\cos\theta$ and $\cos^{-1}\theta$

The inverse trigonometric functions $\cos^{-1} x$ can also be defined by inverting the functions $\cos x$ with domain restricted to $0 \le x \le \pi$.

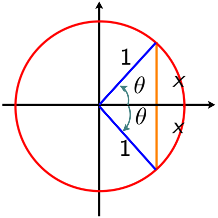




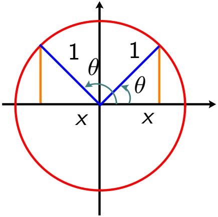
Inverse Trigonometric Functions

Another way to look at these inverse trigonometric functions is to consider solutions of trigonometric equations:

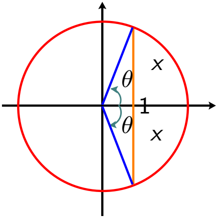
- $\sin^{-1} x$ is the unique solution θ (angle in radian measure) of the equation $x = \sin\theta$ chosen within the closed interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (solvable for any $-1 \leq x \leq 1$).
- $\cos^{-1} x$ is the unique solution θ (angle in radian measure) of the equation $x = \cos\theta$ chosen within the closed interval $[0, \pi]$ (solvable for any $-1 \leq x \leq 1$).
- $\tan^{-1}\theta$ is the unique solution θ (angle in radian measure) of the equation $x = \tan\theta$ chosen within the open interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (solvable for any $-\infty < x < \infty$).



$$-\tfrac{\pi}{2} \le \theta = \sin^{-1} x \le \tfrac{\pi}{2}$$
$$-1 \le x \le 1$$



$$0 \le \theta = \cos^{-1} x \le \tfrac{\pi}{2}$$
$$-1 \le x \le 1$$



$$-\tfrac{\pi}{2} \le \theta = \tan^{-1} x \le \tfrac{\pi}{2}$$
$$-\infty < x < \infty$$

General Solution of Trigonometric Equations

Using the inverse trigonometric functions, one can express the general solutions of some basic trigonometric equations as follows:

$$\sin x = a \quad \begin{cases} x = n\pi + (-1)^n \sin^{-1} a & \text{if} \quad -1 < a < 1 \\ x = 2n\pi + \frac{\pi}{2} & \text{if} \quad a = 1 \\ x = 2n\pi - \frac{\pi}{2} & \text{if} \quad a = -1 \\ \text{no solution} & \text{if} \quad |a| > 1 \end{cases}$$

$$\cos x = a \quad \begin{cases} x = 2n\pi \pm \cos^{-1} a & \text{if} \quad -1 \le a \le 1 \\ \text{no solution} & \text{if} \quad |a| > 1 \end{cases}$$

$$\tan x = a \qquad x = n\pi + \tan^{-1} a \qquad \text{for any real number } a$$

where $n = 0, \pm 1, \pm 2, \pm 3, \cdots$ goes through the set of all integers.

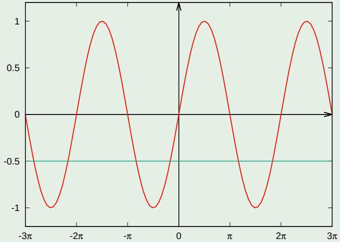
These formulas are based on the fact that the general solutions of trigonometric equations can be found from one known particular solution and periodic properties of trigonometric functions.

Examples of Solving Trigonometric Equations

Example

Find the general solution of the equation $\sin x = -\frac{1}{2}$.

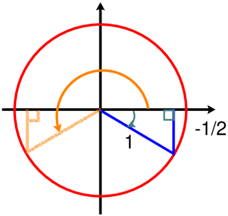
We have $x = n\pi + (-1)^n \sin^{-1}\left(-\frac{1}{2}\right) = n\pi - (-1)^n \frac{\pi}{6}$, where $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$ is a special solution of the equation, $n = 0, \pm 1, \pm 2, \ldots$



Examples of Solving Trigonometric Equations

We can also consider the unit circle:

$\sin(\sin^{-1}(-\frac{1}{2})) = -\frac{1}{2}$



$\pi - \sin^{-1}(-\frac{1}{2})$

Adding integer multiples of the periods to generate all solutions:

$$x = \begin{cases} 2n\pi + \sin^{-1}\left(-\frac{1}{2}\right) \\ 2n\pi + \pi - \sin^{-1}\left(-\frac{1}{2}\right) \\ n = 0, \pm 1, \pm 2, \ldots \end{cases} \iff \begin{cases} x = n\pi + (-1)^n \sin^{-1}\left(-\frac{1}{2}\right) \\ n = 0, \pm 1, \pm 2, \ldots \end{cases}$$

Outline

The Slope of a Tangent Line

In geometry, the *tangent line* to a curve at a given point is the straight line that "just touches" the curve at that point.

The *secant line* of a curve is a line that intersects the curve at a minimum of two distinct points.

Recall that the slope of a straight line passing through two distinct points (x_1, y_1), (x_2, y_2) is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Let's consider the function $y = f(x) = x^2$. How to find the slope $m_{\tan}$ of the tangent line to the graph of f at the point $(1, 1)$?

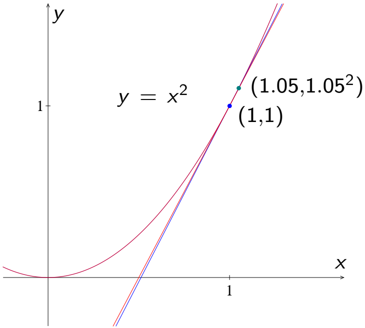
The Slope of a Tangent Line

How to find the slope $m_{\tan}$ of the tangent line to the graph of $y = f(x) = x^2$ at the point $A = (1, 1)$?

In addition to the point $A = (1, 1)$, we can take a nearby point $B = (1.05, 1.05^2)$ on the graph of $y = x^2$.

We can find the slope m_{sec} of the secant line of the graph which passes through these two points:

$$m_{sec} = \frac{1.05^2 - 1}{1.05 - 1} = 2.05 \approx m_{\tan}.$$



The Slope of a Tangent Line

We can get better and better approximation of the slope $m_{\tan}$ by looking at slope of secant line through $(1, 1)$ and another point $(1 + h, (1 + h)^2)$ on the graph, when h is chosen to be closer and closer to 0:

h	0.05	0.005	0.0005	0.00005
m_{sec}	2.05	2.005	2.0005	2.00005

In general, for any $h \neq 0$, we have

$$m_{\text{sec}} = \frac{(1 + h)^2 - 1}{(1 + h) - 1} = \frac{2h + h^2}{h} = 2 + h$$

Note that as $h \to 0$ ("as $h \neq 0$ is approaching 0"), m_{sec} is approaching the number 2, which gives us the slope of the tangent line $m_{\tan} = 2$.

The Slope with Limit Notation

In terms of the "limit notation", this process can be written as

$$\lim_{h \to 0} \frac{(1+h)^2 - 1}{(1+h) - 1} = 2 = m_{\tan}$$

The equation of the tangent line to the graph of $y = x^2$ at the point $(1, 1)$ is then give by

$$\frac{y-1}{x-1} = 2 \Longleftrightarrow y = 2x - 1$$

In fact, it is easy to check that the straight line given by $y = 2x - 1$ intersects the graph of $y = x^2$ at exactly the point $(1, 1)$ by solving the equation

$$x^2 = 2x - 1 \Longleftrightarrow x^2 - 2x + 1 = (x-1)^2 = 0 \Longleftrightarrow x = 1\ .$$

Example (tangent line problem of cubic function)

Find the equation of the tangent line to the curve defined by the equation $y = x^3$ at the point $(2, 8)$.

Consider the slope of the secant line passing through the point $(2, 8)$ and a nearby point $(2 + h, (2 + h)^3)$ on the curve. Then

$$m_{\text{sec}} = \frac{(2+h)^3 - 8}{(2+h) - 2} = \frac{12h + 6h^2 + h^3}{h} = 12 + 6h + h^2 \longrightarrow 12 \quad \text{as } h \to 0$$

Using the limit notation, we have

$$\lim_{h\to 0} \frac{(2+h)^3 - 8}{(2+h) - 2} = \lim_{h\to 0} \frac{12h + 6h^2 + h^3}{h} = \lim_{h\to 0} (12 + 6h + h^2) = 12$$

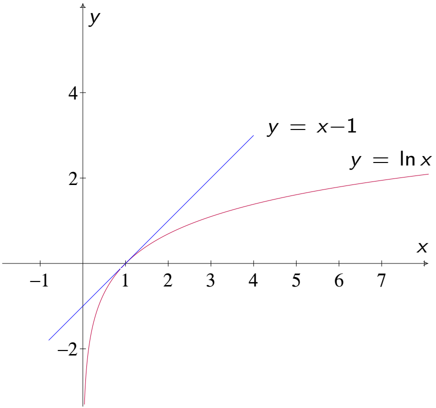
Hence the slope of the tangent line is 12, and the equation of the tangent line to the graph of $y = x^3$ at $(2, 8)$ is

$$\frac{y-8}{x-2} = 12 \Longleftrightarrow y = 12x - 16$$

Outline

Why $y = e^x, y = \log_e x \triangleq \ln x$ with $e \approx 2.7182$?

One condition that determines the number e, which is the *base of the natural logarithmic function*, is that the slope of the tangent line to the graph of the natural logarithmic function $y = \log_e x = \ln x$ at $(1, 0)$ is 1.



Suppose that the slope of the tangent line to the graph of $y = \log_e x$ at the point $(1, 0)$ is 1. We regard e as an unknown number we want to find.

Then the trending behavior of the slope of the secant line passing through the point $(1, 0)$ and a nearby point $(1 + h, \log_e(1 + h))$ on the graph as $h \to 0$ should be

$$\begin{aligned} m_{\text{sec}} =& \frac{\log_e(1+h) - 0}{(1+h) - 1} = \frac{1}{h} \log_e(1+h) \\ =& \log_e(1+h)^{\frac{1}{h}} \longrightarrow 1 \quad \text{as } h \to 0 \end{aligned}$$

Using the limit notation, e is the number which satisfies

$$\lim_{h \to 0} \log_e(1+h)^{\frac{1}{h}} = 1.$$

Since we have $\log_e e = 1$, one way to define the number e is

$$e = \lim_{h \to 0} (1+h)^{\frac{1}{h}}.$$

Let $h = 10^{-10}$, then $(1 + 10^{-10})^{10^{10}} = 2.7182821 \cdots \approx e = 2.7182818 \cdots$

Tangent Line Problem of Exponential Function

Find the slope of the tangent line to the graph of the natural exponential function $y = e^x$ at the point $(0, 1)$.

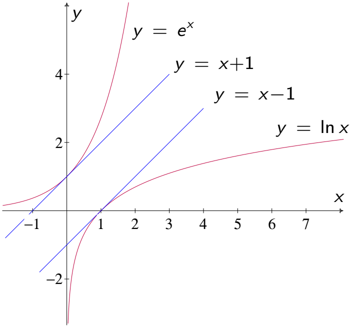
Just recall that the graph of $y = e^x$ can be found by reflecting the graph of its inverse function $y = \ln x$ across the line $y = x$.

The tangent line to the graph of $y = \ln x$ at the point $(1, 0)$ will be reflected to the tangent line to the graph of $y = e^x$ at the point $(0, 1)$.

It is easy to see that the slope of this tangent line to $y = e^x$ is also 1.

Using two nearby points $(0, 1)$, (h, e^h) on the graph of $y = e^x$, and the slope of the secant line passing through them, we have

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1.$$



Tangent Line Problem of Exponential Function

What about the slope of the tangent line to the graph of the natural exponential function $y = e^x$ at the point (a, e^a)?

We have shown that the slope is 1 when $a = 0$. How to extend the result?

Note that the slope of the tangent line to the graph of $y = e^x$ at the point (a, e^a) can then be found by the trending behavior of the slope of the secant line through (a, e^a) and $(a + h, e^{a+h})$

$$\lim_{h \to 0} \frac{e^{a+h} - e^a}{h} = \lim_{h \to 0} \frac{e^a(e^h - 1)}{h} = e^a \cdot \lim_{h \to 0} \frac{e^h - 1}{h} = e^a \cdot 1 = e^a \; .$$

Tangent Line Problem of Sine Function

Find the slope of the tangent line to the graph of the function $y = \sin x$ at the point $(0, 0)$.

By considering the slope of the secant line through the points $(0, 0)$ and $(h, \sin h)$, the slope of the tangent line to the graph of the function $y = \sin x$ at the point $(0, 0)$ is given by

$$\lim_{h\to 0} \frac{\sin h - \sin 0}{h} = \lim_{h\to 0} \frac{\sin h}{h}$$

By calculating a few function values of $\frac{\sin h}{h}$ as $h \to 0$, it is reasonable to *guess* that $\lim_{h\to 0} \dfrac{\sin h}{h} = 1$.

x	0.1	0.01	0.001	0.0001
$\sin h/h$	0.998334166	0.999983333	0.999999833	0.999999998

The slope of the tangent line to the graph of $y = \sin x$ at the origin $(0, 0)$ is then equal to 1. The equation of the tangent line is $y = x$.

Tangent Line Problem of Sine Function

A precise explanation for

$$\lim_{h \to 0} \frac{\sin h}{h} = 1$$

would require more understanding on the limits of function values. We shall look at this limit again later.

Calculus IB: Lecture 05

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Outline

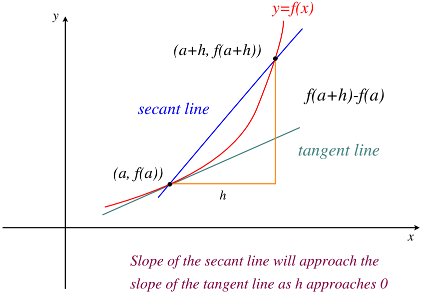
Outline

Limit Definition of Derivative

In general, given a function f, we can consider the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$ in a similar manner by looking at limiting behavior of the slopes of nearby secant lines:

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \triangleq f'(a) \text{ , whenever the limit exists.}$$

$f'(a)$ is called the *derivative of f at a*.



Examples of Derivative

Example

Let $f(x) = \dfrac{1}{x}$. Find the derivative $f'(2)$.

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} = \lim_{h\to 0} \frac{\frac{2-(2+h)}{(2+h)2}}{h} \\ &= \lim_{h\to 0} \frac{-1}{(2+h)2} = \frac{-1}{2\cdot 2} = -\frac{1}{4} \end{aligned}$$

where $f'(2) = -\frac{1}{4}$ can be interpreted as the slope of the tangent line to the graph of $y = \frac{1}{x}$ at the point $(2, \frac{1}{2})$.

Examples of Derivative

The term

$$\frac{f(a+h)-f(a)}{h}$$

is usually considered as the *average rate of change* of the function values of f over the interval $[a, a+h]$, and hence the limit $f'(a)$ is considered as the *instantaneous rate of change* of f at a.

Examples of Derivative

Example

Let $s(t) = t^2$ (in meters) be the position of a particle moving along the s-axis at time t (in seconds).

Consider that $t = 2.05$, then the average rate of change

$$\frac{(1+0.05)^2 - 1}{0.05} = 2.05 \ (\mathrm{m/s})$$

is the usual *average velocity* of the particle on the time interval $[1, 1.05]$.

The instantaneous rate of change of $s = t^2$ at $t = 1$ is called the *instantaneous velocity* of the particle at time $t = 1$:

$$s'(1) = \lim_{h\to 0} \frac{(1+h)^2 - 1}{h} = \lim_{h\to 0} \frac{h(2+h)}{h} = \lim_{h\to 0} 2 + h = 2 \ (\mathrm{m/s})$$

Outline

1. Limit Definition of Derivative
2. Limits of Function Values (Precise Definition)

This section is beyond the requirement of MATH 1013. It will NOT been contained in our homework or exam.

Intuitively speaking, given real numbers c and L, the expression

$$\lim_{x \to c} f(x) = L$$

means that $f(x)$ becomes arbitrarily close to L as x approaches c.

What is "arbitrarily close"? What is "approaches"?

The exact definition of $\lim_{x \to c} f(x)$ is still missing!

Limits of Function Values (Precise Definition)

The phrase "$f(x)$ becomes arbitrarily close to L" means that f(x) eventually lies in the interval $(L-\varepsilon, L+\varepsilon)$, which can also be written as $|f(x)-L|<\varepsilon$.

The phrase "as x approaches c" refer to values of x, whose distance from c is less than some positive number δ that is, values of x within either $(c-\delta, c)$ or $(c, c+\delta)$, which can be expressed with $0<|x-c|<\delta$.

The (ε, δ)-definition of limit: The expression $\lim_{x \to c} f(x) = L$ means for every $\varepsilon>0$ there exists $\delta>0$ such that $|f(x)-L|<\varepsilon$ whenever $0<|x-c|<\delta$ (it does NOT contain $x=c$).

1. It is possible that we cannot find any L such that $\lim_{x \to c} f(x) = L$.
2. We do not require $f(x)$ is well-defined at c, since $x=c$ does not satisfy $0<|x-c|<\delta$.

Example: Show that $\lim_{x \to 2} x^2 = 4$ by the (ε, δ)-definition.

Let $\varepsilon > 0$ be given; we want to show i.e., $|x^2 - 4| < \varepsilon$. How do we find δ such that any x satisfies $|x - 2| < \delta$, we are guaranteed that $|x^2 - 4| < \varepsilon$?

Note that $|x^2 - 4| = |x - 2| \cdot |x + 2|$, hence we consider (suppose $x \neq -2$)

$$|x^2 - 4| < \varepsilon \iff |x - 2| \cdot |x + 2| < \varepsilon \iff |x - 2| < \frac{\varepsilon}{|x + 2|}$$

We cannot set $\delta = \varepsilon / |x + 2|$, because given $\varepsilon > 0$, δ must be a constant value which is independent on x (δ can depends on ε).

Example: Show that $\lim_{x\to 2} x^2 = 4$ by the (ε, δ)-definition.

Recall that limit focus on local property of x^2 at $x = 2$, which implies that δ may be a small value. We can (probably) assume that $\delta < 1$. If this is true, then $|x-2| < \delta$ would imply that $|x-2| < 1$ and $1 < x < 3$, which implies $3 < x+2 < 5$. Then we have

$$\frac{1}{5} < \frac{1}{x+2} < \frac{1}{3} \Longrightarrow \frac{\varepsilon}{5} < \frac{\varepsilon}{x+2}$$

We want to keep $|x-2| < \frac{\varepsilon}{|x+2|}$. Then just take $\delta = \frac{\varepsilon}{5}$, we have

$$|x-2| < \delta \Longrightarrow |x-2| < \frac{\varepsilon}{5} < \frac{\varepsilon}{x+2} \Longrightarrow |x^2-4| < \varepsilon.$$

Recall that we have assume $\delta < 1$ (corresponds to $\varepsilon < 5$), which does not holds if $\varepsilon \geq 5$. Because in such case, we have $\delta = \varepsilon/5 \geq 1$.

Example: Show that $\lim_{x \to 2} x^2 = 4$ by the (ε, δ)-definition.

When $\varepsilon \geq 5$, we can just let $\delta = 1$. Similar to previous analysis for any $|x - 2| < \delta = 1$, we have $-1 < x - 2 < 1$ and $3 < x + 2 < 5$, hence

$$|x^2 - 4| = |x - 2| \cdot |x + 2| < 1 \cdot 5 = 5 \leq \varepsilon \longrightarrow |x^2 - 4| < \varepsilon.$$

In summary, for every $\varepsilon > 0$, there exists $\delta = \min\left(\frac{\varepsilon}{5}, 1\right) > 0$ such that $|x^2 - 4| < \varepsilon$ whenever $0 < |x - 2| < \delta$. Based on the (ε, δ)-definition of limit, we can conclude that

$$\lim_{x \to 2} x^2 = 4.$$

We have provide the precise definition of

$$\lim_{x \to c} f(x) = L$$

when c and L are real numbers.

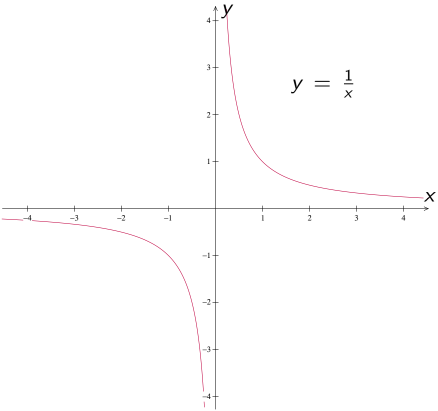
Recall the endpoints of intervals are not limited to real number. For example, we can define $(-\infty, 1]$, $(3, \infty)$, $(-\infty, \infty) \dots$

What about infinity in limits?

Finite Limit at Infinity

Let $f(x) = \frac{1}{x}$. Then $f(x)$ becomes arbitrarily close to 0 as x is arbitrary large. We can write

$$\lim_{x \to \infty} f(x) = 0.$$



Finite Limit at Infinity

More general, given function $f(x)$ and real number L, the expression

$$\lim_{x \to \infty} f(x) = L$$

means for every $\varepsilon > 0$ there exists $N > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > N$.

Similarly, given function $f(x)$ and real number L, the expression

$$\lim_{x \to -\infty} f(x) = L$$

means for every $\varepsilon > 0$ there exists $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.

Finite Limit at Infinity

Example

Let $f(x) = \dfrac{1}{x}$. Show that $\lim_{x\to\infty} f(x) = 0$.

For every $\varepsilon > 0$, there exists $N = \dfrac{1}{\varepsilon}$ such that for any $x > N$, we have

$$|f(x) - 0| = \frac{1}{x} < \frac{1}{N} = \varepsilon.$$

Infinite Limit at Real Number

Given function $f(x)$ and real number c, the expression

$$\lim_{x \to c} f(x) = \infty$$

means for every $M > 0$ there exists $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - c| < \delta$.

Similarly, given function $f(x)$ and real number c, the expression

$$\lim_{x \to c} f(x) = -\infty$$

means for every $M < 0$ there exists $\delta > 0$ such that $f(x) < M$ whenever $0 < |x - c| < \delta$.

Exercise

Let $f(x) = \dfrac{1}{x^2}$. Using above definition to show that $\lim_{x \to 0} f(x) = \infty$.

Note that the result

$$\lim_{x \to c} f(x) = \infty \quad \text{or} \quad \lim_{x \to c} f(x) = -\infty,$$

implies the limit is undefined on real numbers.

However, the condition $\lim_{x \to c} f(x)$ is undefined on real numbers does NOT mean

$$\lim_{x \to c} f(x) = \infty \quad \text{or} \quad \lim_{x \to c} f(x) = -\infty,$$

Can you provide an example? (consider trigonometric functions)

Infinite Limit at Infinity

Given function $f(x)$, the expression

$$\lim_{x\to\infty} f(x) = \infty$$

means for every $M > 0$ there exists $N > 0$ such that $f(x) > M$ whenever $x > N$.

Similarly, we can also give the definitions of

1. $\lim_{x\to\infty} f(x) = -\infty$,
2. $\lim_{x\to-\infty} f(x) = \infty$,
3. $\lim_{x\to-\infty} f(x) = -\infty$.

The (ε, δ)-Definition

We can use (ε, δ)-definition to define a lot of things in calculus such as limit, continuity, derivative and integral.

This section is NOT contained in the requirement of MATH 1013. You can ignore this topic if you think it is difficult.

The (ε, δ)-definition is important to a deeper understanding of calculus. You can read a book of mathematical analysis if you are interested in it, but we will not talk about this topic in our course anymore.

Calculus IB: Lecture 06

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Outline

Outline

Come back the requirement of MATH 1013!

An important point to keep in mind is that finding $\lim_{x \to a} f(x)$ is NOT the same as finding the function value $f(a)$.

1. $\lim_{x \to a} f(x)$ may exist even if $f(x)$ is undefined at $x = a$
2. $\lim_{x \to a} f(x)$ may not exists even if $f(x)$ is well-defined at $x = a$

Examples of Limits of Function Values

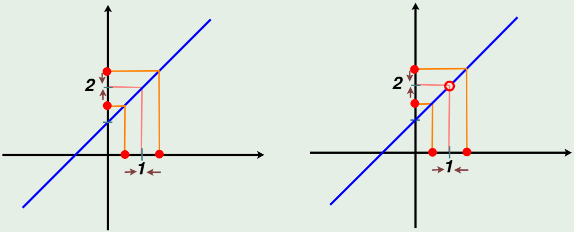
Example

Consider $f(x) = \dfrac{x^2 - 1}{x - 1}$ and $g(x) = x + 1$.

We have $\lim_{x \to 1} g(x) = \lim_{x \to 1} (x + 1) = 1 + 1 = 2 = g(1)$ and

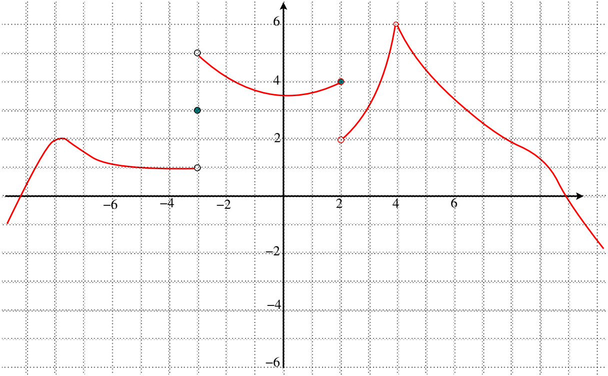
$$\lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} = \lim_{x \to 1} (x + 1) = 1 + 1 = 2,$$

but there is no well-defined function value $f(1)$.

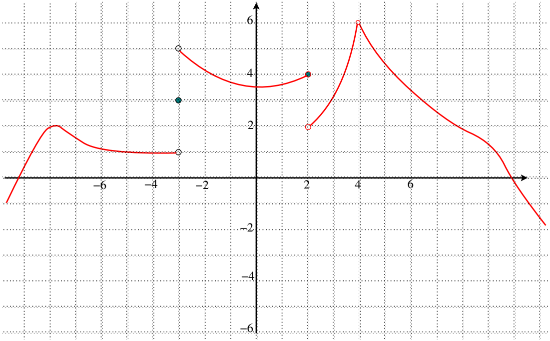


Finding Limits by Graphs

Graphically speaking, finding limits of function values is like riding along the graph (hollow circle means the function value is undefined at this point).



- $f(0)$ is well-defined, and $\lim_{x \to 0} f(x) = f(0)$.
- $\lim_{x \to 4} f(x) = 6$, while $f(4)$ is not well-defined.



- $f(-3) = 3$, but the left-hand limit is

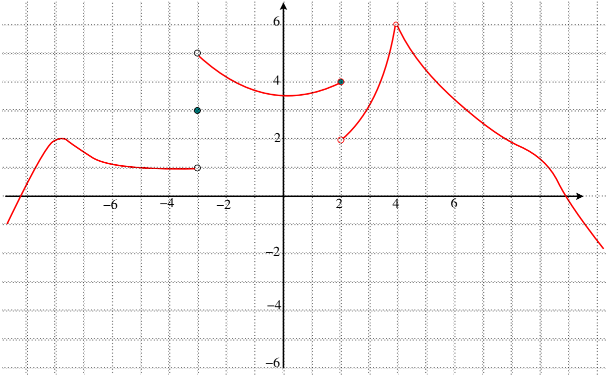
$$\lim_{x \to -3^-} f(x) = 1 \neq f(-3)$$

and the right-hand limit is

$$\lim_{x \to -3^+} f(x) = 5 \neq f(-3)$$

- $x \to -3^-$ means that x is approaching -3 from the left (i.e. $x < -3$)
- $x \to -3^+$ means that x is approaching -3 from the right (i.e. $x > -3$).

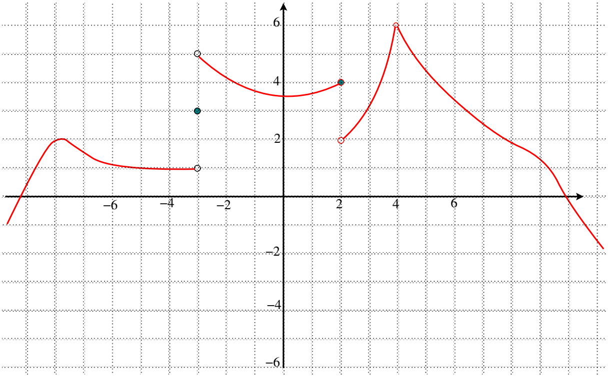
One-Side Limits



- Moreover, $\lim_{x \to -3} f(x)$ does not exist since

$$\lim_{x \to -3^-} f(x) = 1 \neq \lim_{x \to -3^+} f(x) = 5$$

One-Side Limits



- What happens as $x \to 2^-$, or $x \to 2^+$?
- We have $\lim_{x \to 2^-} f(x) = 4 = f(2)$, but $\lim_{x \to 2^+} f(x) = 2 \neq f(2) = 4$.
- The (two-sided) limit $\lim_{x \to 2} f(x)$ does not exist!

Limits and One-Side Limits

The limit $\lim_{x\to a} f(x)$ exists and equals the value L if and only if the two one-sided limits exist, and are equal to L:

$$\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x).$$

Example

Let $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$, then $f(0) = 0$, and

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-1) = -1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1 = 1$$

Since the two one-sided limits are not equal, $\lim_{x\to 0} f(x)$ does not exist.

Exercise

Sketch the graph of the following *piece-wise defined function*

$$f(x) = \begin{cases} x+2 & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ 2x+1 & \text{if } x > 3 \end{cases}$$

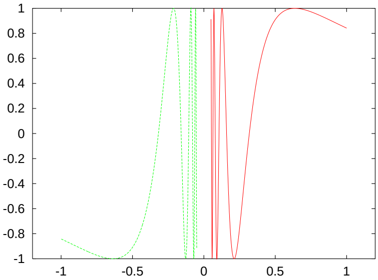
and find the one-sided limits $\lim_{x \to 3^-} f(x)$ and $\lim_{x \to 3^+} f(x)$.

Does $\lim_{x \to 3} f(x)$ exist?

The function $f(x) = \sin\frac{\pi}{x}$ does not have any one-sided limit as $x \to 0^-$ or $x \to 0^+$.

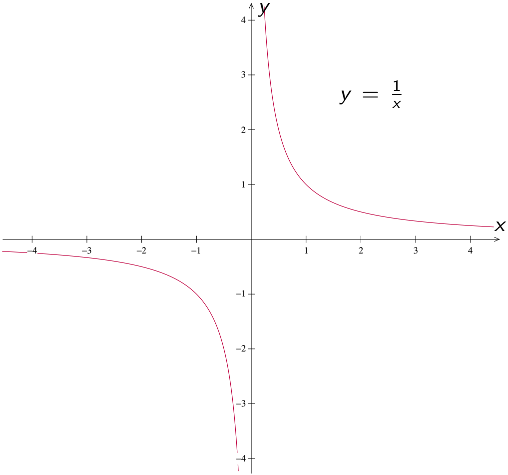
The function value $f(x)$ keeps running up and down through the numbers between -1 and 1 without getting closer and closer to any fixed number when $x \to 0^-$, or $x \to 0^+$.

Note that $\sin\frac{\pi}{x} = 0$ whenever $\frac{\pi}{x} = n\pi$ for some integer n; i.e., whenever $x = \frac{1}{n}$ for some integer $n \neq 0$.



Outline

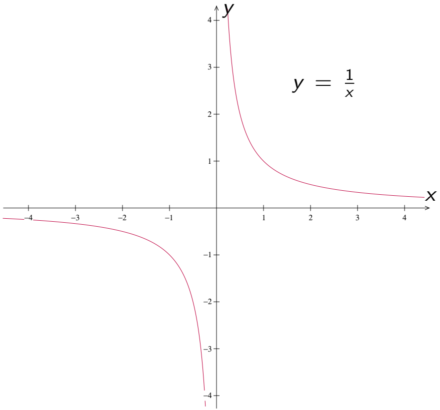
Consider the limit of the function $f(x) = \frac{1}{x}$ as $x \to 0^-, 0^+, -\infty, \infty$ or some constant $a \neq 0$. We can find these limits by graph of $f(x) = \frac{1}{x}$.



Limits of a Function $f(x)$ as $x \to \infty$ or $x \to -\infty$

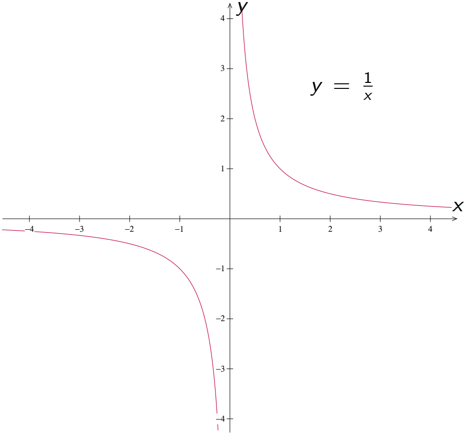
(a) $\displaystyle\lim_{x\to 0^+} \frac{1}{x} = +\infty$ (b) $\displaystyle\lim_{x\to 0^-} \frac{1}{x} = -\infty$ (c) $\displaystyle\lim_{x\to +\infty} \frac{1}{x} = 0$

(d) $\displaystyle\lim_{x\to -\infty} \frac{1}{x} = 0$ (e) $\displaystyle\lim_{x\to a} \frac{1}{x} = \frac{1}{a}$ for all real number $a \neq 0$



Horizontal Asymptote and Vertical Asymptote

The line $y = 0$ (x-axis) is called a *horizontal asymptote* of the function $f(x) = \frac{1}{x}$. The line $x = 0$ (y-axis) is called a *vertical asymptote* of this function.



Horizontal Asymptote and Vertical Asymptote

In general, we may consider the limiting behavior of $f(x)$ as $x \to \infty$ or $x \to -\infty$, or consider some one-sided limits to see if $f(x)$ is approaching ∞ or $-\infty$ as $x \to a^+$ or $a \to a^-$.

1. $y = L$ is a *horizontal asymptote* of the function $f(x)$ if either $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$.
2. $x = b$ is a *vertical asymptote* of the function $f(x)$ if at least one of the following holds:
    1. $\lim\limits_{x\to b^-} f(x) = \infty$
    2. $\lim\limits_{x\to b^-} f(x) = -\infty$
    3. $\lim\limits_{x\to b^+} f(x) = \infty$
    4. $\lim\limits_{x\to b^+} f(x) = -\infty$

Note that f has two different horizontal asymptotes $y = L_1$ and $y = L_2$ if

$$\lim_{x\to\infty} f(x) = L_1 \neq \lim_{x\to-\infty} f(x) = L_2$$

In any case, a function can have at most two horizontal asymptotes.

Example

Find horizontal asymptote and vertical asymptote the function $f(x) = \dfrac{1}{x-2}$ by running along its graph.

(a) $\displaystyle\lim_{x\to+\infty} \frac{1}{x-2} = 0$

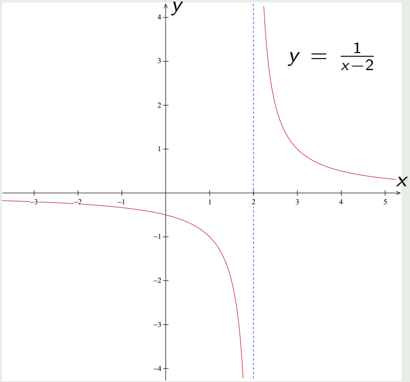
(b) $\displaystyle\lim_{x\to-\infty} \frac{1}{x-2} = 0$

(c) $\displaystyle\lim_{x\to 2^+} \frac{1}{x-2} = +\infty$

(d) $\displaystyle\lim_{x\to 2^-} \frac{1}{x-2} = -\infty$

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = 2$



Horizontal Asymptote and Vertical Asymptote

Example

Find horizontal asymptote and vertical asymptote the function $f(x) = \dfrac{x-1}{x} = 1 - \dfrac{1}{x}$ by running along its graph.

(a) $\displaystyle\lim_{x\to+\infty} \frac{x-1}{x} = 1$

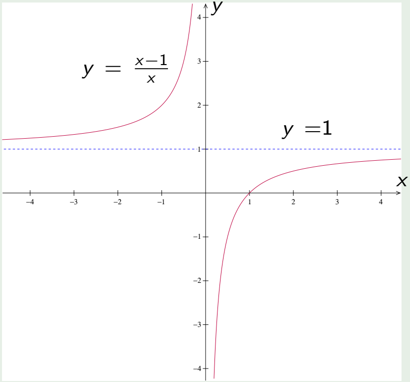
(b) $\displaystyle\lim_{x\to-\infty} \frac{x-1}{x} = 1$

(c) $\displaystyle\lim_{x\to 0^+} \frac{x-1}{x} = -\infty$

(d) $\displaystyle\lim_{x\to 0^-} \frac{x-1}{x} = +\infty$

Horizontal asymptote: $y = 1$

Vertical asymptote: $x = 0$



Vertical Asymptote and Slant Asymptote

Summary of above results:

1. Given a function of the form $\dfrac{f(x)}{g(x)}$, the vertical line defined by $x = a$ is a vertical asymptote as long as $f(a) \neq 0$ (**Correction**: we do NOT require $f(x)$ is well-defined at a, but $f(a)$ cannot be 0 if it is well-defined) but $\lim_{x \to a^-} g(x) = 0$ or $\lim_{x \to a^+} g(x) = 0$.
2. If $f(x) = ax + b + g(x)$ with $g(x) \to 0$ as $x \to \infty$ or $x \to -\infty$, then the straightline given by $y = ax + b$ is called a *slant asymptote* of f.

Example of Slant Asymptote

Consider function $f(x) = \dfrac{x^2 + 2x + 3}{x} = x + 2 + \dfrac{3}{x}$.

(a) $x = 0$ is a vertical asymptote of f since

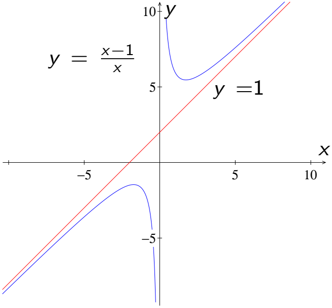
$$\lim_{x \to 0^+} \left(x + 2 + \frac{3}{x}\right) = \infty$$

$$\lim_{x \to 0^-} \left(x + 2 + \frac{3}{x}\right) = -\infty$$

(b) $y = x + 2$ is a slant asymptote of f since

$$\lim_{x \to \infty} (f(x) - (x + 2)) = \lim_{x \to \infty} \frac{3}{x} = 0$$

$$\lim_{x \to -\infty} (f(x) - (x + 2)) = \lim_{x \to -\infty} \frac{3}{x} = 0$$



Exercise of Slant Asymptote

Exercise

Show that $y = -x$ and $y = x$ are two slant asymptotes of the function $f(x) = \sqrt{1+x^2}$.

Hint: Consider the values of

$$\lim_{x\to\infty}\left(\sqrt{1+x^2}-x\right) = \lim_{x\to\infty}\left(\frac{(\sqrt{1+x^2}-x)(\sqrt{1+x^2}+x)}{\sqrt{1+x^2}+x}\right)$$

and

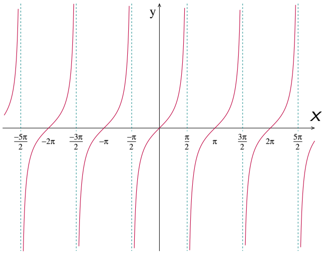
$$\lim_{x\to-\infty}\left(\sqrt{1+x^2}+x\right) = \lim_{x\to-\infty}\left(\frac{(\sqrt{1+x^2}+x)(\sqrt{1+x^2}-x)}{\sqrt{1+x^2}-x}\right).$$

Examples of Multiple Vertical Asymptotes

Consider the function $f(x) = \tan(x)$, we have

$$\lim_{x \to a^+} \tan(x) = -\infty \quad \text{and} \quad \lim_{x \to a^-} \tan(x) = \infty,$$

where $a = \dfrac{\pi}{2} + n\pi$ for any integer n.

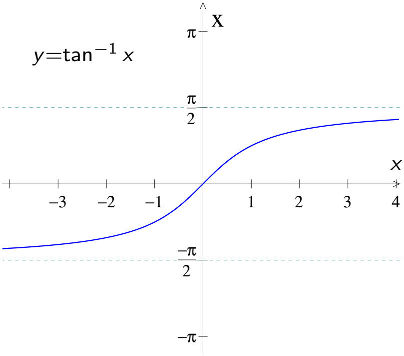


Hence $x = \dfrac{\pi}{2} + n\pi$ with any integer n is a vertical asymptote and there are infinite asymptotes in total.

Examples of Multiple Horizontal Asymptotes

Consider the function $f(x) = \tan^{-1}(x)$, we have

$$\lim_{x\to\infty} \tan(x) = \frac{\pi}{2} \quad \text{and} \quad \lim_{x\to-\infty} \tan(x) = -\frac{\pi}{2}.$$

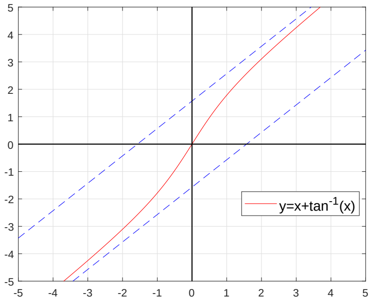


Hence $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are two horizontal asymptotes.

Examples of Multiple Slant Asymptotes

Consider the function $f(x) = x + \tan^{-1}(x)$, we have

$$\lim_{x\to\infty}\left(f(x) - x - \frac{\pi}{2}\right) = 0 \quad \text{and} \quad \lim_{x\to-\infty}\left(f(x) - x + \frac{\pi}{2}\right) = 0.$$



Hence $y = x - \dfrac{\pi}{2}$ and $y = x + \dfrac{\pi}{2}$ are two slant asymptotes.

Outline

Some Useful Limit Laws

Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exists on *real numbers*, then we have:

1. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$ for any constant c
2. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}[f(x)]^p = \left(\lim_{x\to a} f(x)\right)^p$ for any rational exponent p when $(\lim_{x\to a} f(x))^p$ exists.

Some Useful Limit Laws

All of these rules can be proved by precise definition of limit, which is based on (ε, δ) language.

The following things are undefined:

$$\frac{\infty}{\infty}, \quad \frac{0}{0}, \quad 0 \cdot \infty \quad \text{and} \quad \infty - \infty.$$

Some Useful Limit Laws

Let $f(x) = \frac{1}{x^2}$ and $g(x) = -\frac{1}{x^2}$. What is $\lim_{x \to a}[f(x) + g(x)]$?

The definition of $f(x)$ and $g(x)$ means

$$f(x) + g(x) = \begin{cases} 0, & x \neq 0, \\ \text{undefined}, & x = 0. \end{cases}$$

Then we have $\lim_{x \to 0^+}[f(x) + g(x)] = \lim_{x \to 0^-}[f(x) + g(x)] = 0$ and

$$\lim_{x \to 0}[f(x) + g(x)] = 0.$$

However, we have

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{1}{x^2} = \infty \quad \text{and} \quad \lim_{x \to 0} g(x) = \lim_{x \to 0} -\frac{1}{x^2} = -\infty.$$

Since $\infty + (-\infty)$ is undefined, we can NOT say $\infty + (-\infty) = 0$

Examples of Limits

After checking the existence of limit, we can use above rules.

Example

Find $\lim_{x\to 2}(x^2 - 2x + 5)$ and $\lim_{x\to 2} \sqrt[3]{x^2 - 2}$.

$$
\begin{aligned}
&\lim_{x\to 2}(x^2 - 2x + 5) \\
=&\left(\lim_{x\to 2} x\right)^2 - 2\lim_{x\to 2} x + \lim_{x\to 2} 5 \\
=&2^2 - 2\cdot 2 + 5 = 7
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{x\to 2} \sqrt[3]{x^2 - 2} \\
=&\sqrt[3]{\lim_{x\to 2}(x^2 - 2)} \\
=&\sqrt[3]{2^2 - 2} = \sqrt[3]{2}
\end{aligned}
$$

Example

Find $\lim\limits_{x\to 2} \dfrac{2x^2 - x + 1}{x^2 - 1}$.

$$\begin{aligned}
&\lim_{x\to 2} \frac{2x^2 - x + 1}{x^2 - 1} \\
=&\frac{\lim\limits_{x\to 2} \left(2x^2 - x + 1\right)}{\lim_{x\to 2}(x^2 - 1)} \\
=&\frac{2\cdot 2^2 - 2 + 1}{2^2 - 1} = \frac{7}{3}
\end{aligned}$$

Examples of Limits

Several algebraic tricks, mostly about factor canceling, are often needed in order to find limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example

Find the limit $\lim_{x\to 2} \dfrac{x^2-4}{x-2}$. $\left(\dfrac{0}{0}\text{-type limit}\right)$

Note that if we directly substitute $x = 2$ into the expression, we will get some undefined expression $\frac{0}{0}$. This suggests that $(x-2)$ is a factor of both the numerator and the denominator. After factoring, we have

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2}(x+2) = 2+2 = 4$$

Note that for $x \to 2$, we do not need to consider $x = 2$.

Examples of Limits

Example

Find the limit $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$.

It is also a $\frac{0}{0}$ type limit. By suitable factor cancellation, we have

$$
\begin{aligned}
&\lim_{x\to 9} \frac{\sqrt{x}-3}{x-9} \\
=&\lim_{x\to 9} \frac{\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+3)} \\
=&\lim_{x\to 9} \frac{1}{\sqrt{x}+3} \\
=&\frac{1}{\sqrt{9}+3} = \frac{1}{6}.
\end{aligned}
$$

More Examples of $\frac{0}{0}$ Type Limits

Here are some more examples of $\frac{0}{0}$ Type Limits, found by algebraic transformation.

Example

Find $\displaystyle\lim_{x\to 0} \frac{\sqrt{2x+1}-1}{x}$.

$$\begin{aligned}
&\lim_{x\to 0} \frac{\sqrt{2x+1}-1}{x} \\
=&\lim_{x\to 0} \frac{\sqrt{2x+1}-1}{x} \cdot \frac{\sqrt{2x+1}+1}{\sqrt{2x+1}+1} \\
=&\lim_{x\to 0} \frac{2x}{x(\sqrt{2x+1}+1)} \\
=&\lim_{x\to 0} \frac{2}{\sqrt{2x+1}+1} = 1
\end{aligned}$$

More Examples of $\frac{0}{0}$ Type Limits

Example

Find $\lim\limits_{x \to 0} \dfrac{x^2}{\sqrt{x^2+4}-2}$.

$$\begin{aligned}
&\lim_{x \to 0} \frac{x^2}{\sqrt{x^2+4}-2} \\
=&\lim_{x \to 0} \frac{x^2}{\sqrt{x^2+4}-2} \cdot \frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2} \\
=&\lim_{x \to 0} \frac{x^2(\sqrt{x^2+4}+2)}{x^2} \\
=&4
\end{aligned}$$

Calculus IB: Lecture 07

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Outline

Outline

1. Basic Techniques in Limit Computation (Cont'd)
2. Extended Real Number System
3. Squeeze Theorem

Example

Find the limit $\lim\limits_{x\to+\infty} \dfrac{2x^2 - x + 3}{3x^2 + x - 1}$.

We need to understand the behavior of the function $\frac{1}{x}$ as $x \to +\infty$:

$$\lim_{x\to+\infty} \frac{2x^2 - x + 3}{3x^2 + x - 1} = \lim_{x\to+\infty} \frac{x^2(2 - \frac{1}{x} + \frac{3}{x^2})}{x^2(3 + \frac{1}{x} - \frac{1}{x^2})}$$

$$= \lim_{x\to+\infty} \frac{2 - \frac{1}{x} + \frac{3}{x^2}}{3 + \frac{1}{x} - \frac{1}{x^2}} = \frac{2 - 0 + 3\cdot 0}{3 + 0 - 0} = \frac{2}{3},$$

where we use the fact

$$\lim_{x\to+\infty} \frac{1}{x} = 0, \quad \lim_{x\to+\infty} \frac{1}{x^2} = \lim_{x\to+\infty} \frac{1}{x} \cdot \lim_{x\to+\infty} \frac{1}{x} = 0 \cdot 0 = 0.$$

Examples of $\frac{\infty}{\infty}$ Type Limits

Example

(a) $$\lim_{x\to+\infty} \frac{\sqrt{2x+1}-1}{x} = \lim_{x\to+\infty} \left[\sqrt{\frac{2}{x}-\frac{1}{x^2}}-\frac{1}{x}\right] = 0.$$

(b) $$\lim_{x\to+\infty} \frac{x^2}{\sqrt{x^2+4}-2} = \lim_{x\to+\infty} \frac{x}{\sqrt{1+\frac{4}{x^2}}-\frac{2}{x}} = +\infty$$

(c) $$\lim_{x\to+\infty} \frac{2x}{\sqrt{x^2+4}-2} = \lim_{x\to+\infty} \frac{2x}{x(\sqrt{1+\frac{4}{x^2}}-\frac{2}{x})} = \lim_{x\to+\infty} \frac{2}{\sqrt{1+\frac{4}{x^2}}-\frac{2}{x}} = 2$$

Examples of $\infty - \infty$ Type Limits

Example

$$\begin{aligned}
&\lim_{x \to +\infty} (\sqrt{x+1} - \sqrt{x}) \\
= &\lim_{x \to +\infty} \frac{(\sqrt{x+1} - \sqrt{x})(\sqrt{x+1} + \sqrt{x})}{\sqrt{x+1} + \sqrt{x}} \\
= &\lim_{x \to +\infty} \frac{1}{\sqrt{x+1} + \sqrt{x}} \\
= &\ 0
\end{aligned}$$

$$\text{Why } \lim_{x \to +\infty} \frac{1}{\sqrt{x+1} + \sqrt{x}} = 0?$$

Examples of $\infty - \infty$ Type Limits

Example

$$\begin{aligned}
&\lim_{x\to+\infty}\left(\sqrt{x^2+x}-x\right)\\
&=\lim_{x\to+\infty}\frac{\left(\sqrt{x^2+x}-x\right)\left(\sqrt{x^2+x}+x\right)}{\left(\sqrt{x^2+x}+x\right)}\\
&=\lim_{x\to+\infty}\frac{x}{\left(\sqrt{x^2+x}+x\right)}\\
&=\lim_{x\to+\infty}\frac{1}{\sqrt{1+\frac{1}{x}}+1}=\frac{1}{2}
\end{aligned}$$

Examples of $\infty - \infty$ Type Limits

When computing limits of the form

$$\lim_{x \to \infty} (f(x) - g(x)),$$

where both f and g are approaching ∞ as x is approaching ∞, one is actually looking at the trending behaviour of the gap between the graph of f and g, i.e., how

$$f(x) - g(x)$$

behaves as $x \to \infty$.

Examples of $\infty - \infty$ Type Limits

Example

Find one-sided limits: $\lim_{x\to 1^-} \dfrac{x^2 - x + 1}{x^2 - 1}$ and $\lim_{x\to 1^+} \dfrac{x^2 - x + 1}{x^2 - 1}$. We have

$$\lim_{x\to 1^-} \frac{x^2 - x + 1}{x^2 - 1} = \lim_{x\to 1^-} \frac{x^2 - x + 1}{x + 1} \cdot \frac{1}{x - 1} = \frac{1}{2} \cdot (-\infty) = -\infty$$

and

$$\lim_{x\to 1^+} \frac{x^2 - x + 1}{x^2 - 1} = \lim_{x\to 1^+} \frac{x^2 - x + 1}{x + 1} \cdot \frac{1}{x - 1} = \frac{1}{2} \cdot \infty = \infty$$

Hence $x = 1$ is a vertical asymptote of the function $\dfrac{x^2 - x + 1}{x^2 - 1}$.

Outline

Limit Laws with Infinity

We define the arithmetic operations as follows

$$c \cdot \infty = \infty \cdot c = \infty \quad \text{for real number } c > 0.$$

Based on above notations, we can generalize

$$\lim_{x \to a} [f(x)g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$$

to the case that one of $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ is ∞.

Formally speaking, the notation $\infty \cdot c$ means we have

$$\lim_{x \to a} [f(x)g(x)] = \infty$$

when $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = c > 0$.

The Proof of $\infty \cdot c = \infty$

Proof: The limit $\lim_{x \to a} f(x) = \infty$ means for any $M_0 > 0$, there exists $\delta_0 > 0$ such that $f(x) > M_0$ whenever $0 < |x - a| < \delta_0$. Hence, for every $M > 0$, let $M_0 = \frac{2M}{c}$, there exists $\delta_0 > 0$ such that

$$|f(x)| = f(x) > \frac{2M}{c}$$

whenever $0 < |x - a| < \delta_0$.

The limit $\lim_{x \to a} g(x) = c$ means for every $\varepsilon_1 > 0$ there exists $\delta_1 > 0$ such that $|g(x) - c| < \varepsilon_1$ whenever $0 < |x - a| < \delta_1$. Let $\varepsilon_1 = \frac{c}{2}$, there exist $\delta_1 > 0$ such that we have

$$-\frac{c}{2} < g(x) - c < \frac{c}{2} \Longrightarrow \frac{c}{2} < g(x) < \frac{3c}{2} \Longrightarrow |g(x)| > \frac{c}{2}$$

whenever $0 < |x - a| < \delta_1$.

The Proof of $\infty \cdot c = \infty$

For every $M > 0$, there exists $\delta = \min(\delta_0, \delta_1)$ such that

$$|f(x)| > \frac{2M}{c} \text{ and } |g(x)| > \frac{c}{2} \Longrightarrow |f(x)g(x)| > M$$

whenever $0 < |x - a| < \delta$. The notation $\min(\delta_0, \delta_1)$ means the minimizer of δ_0 and δ_1.

Hence we can conclude

$$\lim_{x \to a}[f(x)g(x)] = \infty$$

when $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = c > 0$ by precise definition of limit.

Extended Real Number System

We introduce extended real number system to address the calculation contains the ∞ and $-\infty$. It is useful in describing the algebra on infinities and the various limiting behaviors in calculus.

Recall that $\mathbb{R} = (-\infty, \infty)$ presents the set of all real number.

The extended real number system is denoted by $\overline{\mathbb{R}}$ or $[-\infty, +\infty]$ or $\mathbb{R} \cup \{-\infty, +\infty\}$.

Here, "$+\infty$" is equivalent to "∞" and "$-(-\infty)$".

Arithmetic Operations on $\overline{\mathbb{R}}$

$$
\begin{aligned}
a + \infty = +\infty + a &= +\infty, & a &\neq -\infty \\
a - \infty = -\infty + a &= -\infty, & a &\neq +\infty \\
a \cdot (+\infty) = +\infty \cdot a &= +\infty, & a &\in (0, +\infty] \\
a \cdot (-\infty) = -\infty \cdot a &= -\infty, & a &\in (0, +\infty] \\
a \cdot (+\infty) = +\infty \cdot a &= -\infty, & a &\in [-\infty, 0) \\
a \cdot (-\infty) = -\infty \cdot a &= +\infty, & a &\in [-\infty, 0)
\end{aligned}
$$

$$\frac{a}{+\infty} = \frac{a}{-\infty} = 0, \qquad a \in \mathbb{R}$$

$$\frac{+\infty}{a} = +\infty, \qquad a \in (0, +\infty)$$

$$\frac{-\infty}{a} = -\infty, \qquad a \in (0, +\infty)$$

$$\frac{+\infty}{a} = -\infty, \qquad a \in (-\infty, 0)$$

$$\frac{-\infty}{a} = +\infty, \qquad a \in (-\infty, 0)$$

Arithmetic Operations on $\overline{\mathbb{R}}$

$$
\begin{aligned}
a^{+\infty} &= +\infty & a &\in (1, +\infty] \\
a^{-\infty} &= 0 & a &\in (1, +\infty] \\
a^{+\infty} &= 0 & a &\in [0, 1) \\
a^{-\infty} &= +\infty & a &\in [0, 1) \\
0^{a} &= 0 & a &\in (0, +\infty] \\
(+\infty)^{a} &= +\infty & a &\in (0, +\infty] \\
(+\infty)^{a} &= 0 & a &\in [-\infty, 0)
\end{aligned}
$$

Correction: The rule $0^a = +\infty$ for $a \in [-\infty, 0)$ is **NOT** allowed in MATH 1013, just like $1/0$ we explain in next page.

Extended Real Number System

However, the following expressions are still undefined

$$\frac{+\infty}{+\infty} \qquad \frac{+\infty}{-\infty} \qquad \frac{-\infty}{+\infty} \qquad \frac{-\infty}{-\infty}$$

$$0 \cdot (+\infty) \qquad 0 \cdot (-\infty) \qquad (+\infty) \cdot 0 \qquad (-\infty) \cdot 0$$

$$\infty - \infty \qquad (-\infty) - (-\infty)$$

In the context of probability or measure theory, the product of 0 and ∞ (or $-\infty$) is often defined as 0, but it is **NOT** allowed in MATH 1013.

The expression $1/0$ (or $0^a = +\infty$ for $a \in [-\infty, 0)$) is still left undefined, since

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty \neq -\infty = \lim_{x \to 0^-} \frac{1}{x}.$$

In contexts only non-negative values are considered, it is often convenient to define $1/0 = +\infty$. But it is **NOT** allowed in MATH 1013.

Extended Real Number System

We can extend following laws to extended real number system if all expressions are defined on $\overline{\mathbb{R}}$ based on above slides.

1. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x)$ for any constant c
2. $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$
3. $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$
4. $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
5. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$
6. $\lim_{x\to a}[f(x)]^p = \left(\lim_{x\to a} f(x)\right)^p$ for any rational exponent p when $\left(\lim_{x\to a} f(x)\right)^p$ exists.

Exercise

Find (i) $\lim_{x \to -1^-} \frac{x^2 - x + 1}{x^2 - 1}$, (ii) $\lim_{x \to -1^+} \frac{x^2 - x + 1}{x^2 - 1}$. Can you find all vertical asymptotes of the function $\frac{x^2 - x + 1}{x^2 - 1}$? and any horizontal asymptotes?

Exercise

Compute the limit (a) $\lim_{x \to 0^+} \left(\frac{1}{\sqrt{x}} - \frac{1}{x} \right)$, (b) $\lim_{x \to 2} \sqrt{\frac{x^2 - 5x + 6}{x^2 - 4}}$.

Exercise

Compute the limit (a) $\lim_{x \to e^2} \frac{(\ln x)^3 - 8}{(\ln x)^2 - 4}$, (b) $\lim_{x \to 0} \frac{1 + \sin x}{\cos^2 x}$.

Outline

Squeeze Theorem

Squeeze Theorem (or Sandwich Theorem)

Let I be an interval having the point a. Let g, f, and h be functions defined on I, except possibly at a itself. Suppose that for every x in I NOT equal to a, we have If $g(x) \leq f(x) \leq h(x)$ for all x near a, except perhaps when $x = a$, then

$$\lim_{x \to a} g(x) \leq \lim_{x \to a} f(x) \leq \lim_{x \to a} h(x)$$

whenever these limits exist. (The same is true for one-sided limits.)

Exercise

Try to prove squeeze theorem by (ε, δ)-definition.

Note that we only require $g(x) \leq f(x) \leq h(x)$ holds locally.

Squeeze Theorem

Example

Suppose $1 - 2x^2 \leq f(x) \leq 1 + 3x^2$ for $-1 < x < 1$. Then by the Squeeze Theorem, we have

$$1 = \lim_{x \to 0}(1 - 2x^2) \leq \lim_{x \to 0} f(x) \leq \lim_{x \to 0}(1 + 3x^2) = 1$$

and hence

$$\lim_{x \to 0} f(x) = 1 \ .$$

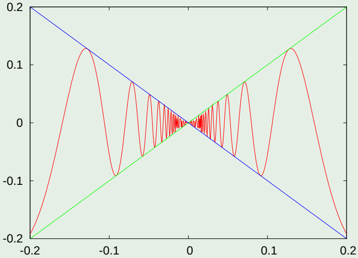
Squeeze Theorem

Example

Show that $\lim_{x\to 0} x\sin\frac{1}{x} = 0$ by applying the Squeeze Theorem:

$$-|x| \le x\sin\frac{1}{x} \le |x|$$

$$0 = -\lim_{x\to 0}|x| \le \lim_{x\to 0} x\sin\frac{1}{x} \le \lim_{x\to 0}|x| = 0\ .$$



Squeeze Theorem

Note that we CANNOT apply the limit law about product to write

$$\lim_{x\to 0} x \sin\frac{1}{x} = \lim_{x\to 0} x \cdot \lim_{x\to 0} \sin\frac{1}{x}$$

since $\lim_{x\to 0} \sin\frac{1}{x}$ does not exist! (neither $+\infty$ nor $-\infty$)

Example

Show that $\lim_{t \to +\infty} e^{-t/2} \sin 5t = 0$.

Since $|\sin 5t| \leq 1$, we have

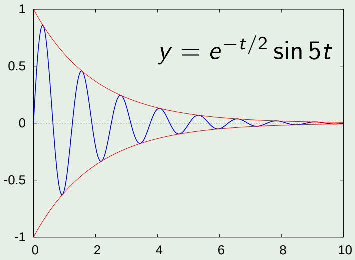
$$-e^{-t/2} \leq e^{-t/2} \sin 5t \leq e^{-t/2}.$$

On the other hand, we have

$$\lim_{t \to +\infty} e^{-t/2} = \lim_{t \to +\infty} -e^{-t/2} = 0.$$

Applying the Squeeze Theorem, then

$$\lim_{t \to +\infty} e^{-t/2} \sin 5t = 0.$$



In Lecture 04, we guessed

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$$

by calculating several points of θ near 0:

θ	0.1	0.01	0.001	0.0001
$\sin\theta/\theta$	0.998334166	0.999983333	0.999999833	0.999999998

Now, we can prove it by Squeeze Theorem.

Examples of Squeeze Theorem

By the Squeeze Theorem, this limit follows easily from the following inequalities: for $0 < \theta < \frac{\pi}{2}$,

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Hence

$$1 = \lim_{\theta\to 0^+} \cos\theta \ \le \ \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} \ \le \ \lim_{\theta\to 0^+} 1 \ = \ 1$$

$$\lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1$$

Note that $\dfrac{\sin\theta}{\theta}$ is an even functions. Hence

$$\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} = 1 \ .$$

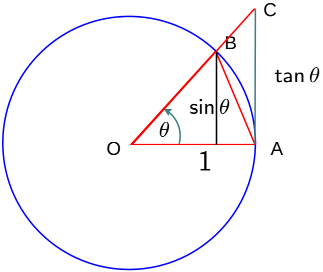
Examples of Squeeze Theorem

To prove $\cos\theta < \frac{\sin\theta}{\theta} < 1$ for $0 < \theta < \pi/2$, we compare the areas of following triangles and circular sector within unit circle:

Area of $\triangle OAB$ < Area of circular sector OAB < Area of $\triangle OAC$

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta = \frac{\sin\theta}{2\cos\theta}$$

$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$



Examples of Squeeze Theorem

Example

Using $\lim_{\theta \to 0} \frac{\sin k\theta}{k\theta} = 1$ for any non-zero constant k, we have

(i) $\lim_{\theta \to 0} \frac{\tan 2\theta}{\theta} = \lim_{\theta \to 0} \left[\frac{\sin 2\theta}{2\theta} \cdot \frac{2}{\cos 2\theta} \right] = \lim_{\theta \to 0} \frac{\sin 2\theta}{2\theta} \cdot \lim_{\theta \to 0} \frac{2}{\cos 2\theta} = 1 \cdot 2 = 2$

(ii) $\lim_{x \to 0} \frac{\sin 3x}{2x} = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \frac{3}{2} = \frac{3}{2}$

(iii) $\lim_{h \to 0} \frac{\cos h - 1}{h} = \lim_{h \to 0} \frac{-2\sin^2 \frac{h}{2}}{h} = -\lim_{h \to 0} \sin \frac{h}{2} \cdot \lim_{h \to 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} = 0 \cdot 1 = 0$

(we use $\cos h = 1 - 2\sin^2(\frac{h}{2})$)

We can prove $\lim_{\theta \to 0} \frac{\sin k\theta}{k\theta} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$ by (ε, δ)-definition.

Examples of Squeeze Theorem

Example

Given the inequality $e^x \geq x + 1$ for all x, show that $\lim\limits_{x\to\infty} \dfrac{x}{e^x} = 0$.

Noting that $e^x = \left(e^{x/2}\right)^2 \geq \left(\dfrac{x}{2} + 1\right)^2 = \dfrac{x^2}{4} + x + 1$, we have for $x > 0$ the inequalities

$$0 < \frac{x}{e^x} \leq \frac{x}{\frac{x^2}{4} + x + 1}$$

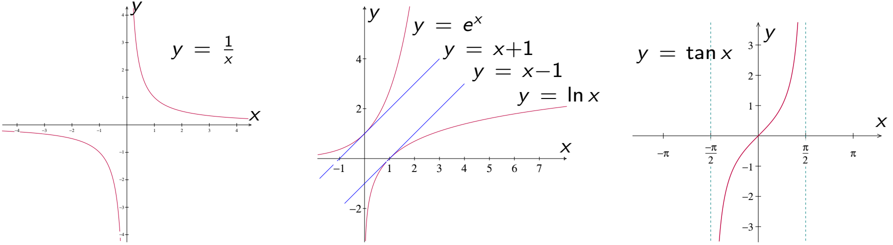
$$0 \leq \lim_{x\to\infty} \frac{x}{e^x} \leq \lim_{x\to\infty} \frac{x}{\frac{x^2}{4} + x + 1} = \lim_{x\to\infty} \frac{1}{\frac{x}{4} + 1 + \frac{1}{x}} = 0$$

$$\lim_{x\to\infty} \frac{x}{e^x} = 0 \ .$$

More generally, $\lim\limits_{x\to\infty} \dfrac{x^n}{e^x} = 0$ for any positive integer n.

Why $e^x \geq x + 1$ holds for all x?

Summary of Some Basic Limits



$$\lim_{x\to\infty} \frac{1}{x} = 0 \qquad \lim_{x\to\infty} e^x = \infty \qquad \lim_{x\to\frac{\pi}{2}^-} \tan x = \infty$$

$$\lim_{x\to-\infty} \frac{1}{x} = 0 \qquad \lim_{x\to-\infty} e^x = 0 \qquad \lim_{x\to-\frac{\pi}{2}^+} \tan x = -\infty$$

$$\lim_{x\to0^+} \frac{1}{x} = \infty \qquad \lim_{x\to\infty} \ln x = \infty \qquad \lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$$

$$\lim_{x\to0^-} \frac{1}{x} = -\infty \qquad \lim_{x\to0^+} \ln x = -\infty \qquad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

Exercises of Squeeze Theorem

Exercise

Show that $\lim_{x \to 0^+} x \ln x = 0$ by letting $x = e^{-t}$.

Exercise

Show that $\lim_{x \to \infty} \frac{x^2}{e^x} = 0$, and hence $\lim_{t \to \infty} \frac{(\ln t)^2}{t} = 0$ by letting $x = \ln t$.

Calculus IB: Lecture 08

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Outline

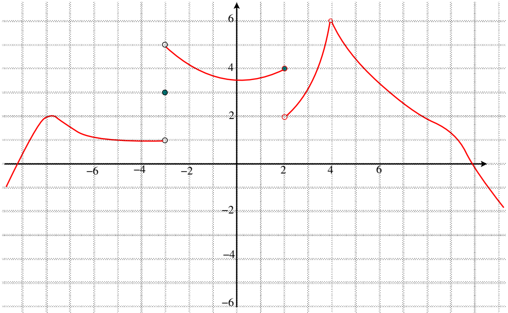
Outline

Continuity of Functions

We have seen that even when $f(c)$, $\lim_{x\to c^-} f(x)$ and $\lim_{x\to c^+} f(x)$ all exist, it is still possible that they are not equal.

When they are all well-defined and equal on real numbers (we do not consider ∞ or $-\infty$), we say that the function is *continuous* at $x = c$.

Roughly speaking, this is a mathematical way to say that no "sudden jump" on the graph will occur when passing through $x = c$.



Continuity of Functions

In this course, we focus on the continuity of functions defined on an interval, or the union of several intervals.

- If c is a real number in the domain of a function f such that a small open interval $(c-h, c+h)$ containing c, where $h > 0$, is entirely in the domain of f, c is called an *interior point* of the domain of f.

  A function $y = f(x)$ is said to be *continuous at an interior point* c in its domain if $\lim_{x \to c} f(x) = f(c)$.

Continuity of Functions

- If a is a number in the domain of of f which is not an interior point, then the continuity condition $\lim_{x\to a} f(x) = f(a)$ should be understood as $f(x)$ is getting closer and closer to $f(a)$ as x in the domain of f is getting closer to closer to a.

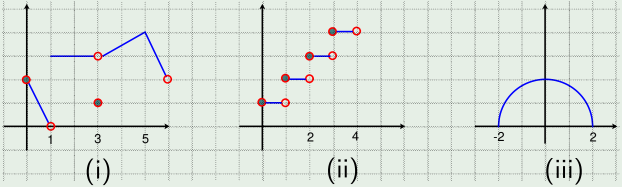
  In particular, $x \to a$ should be understood as $x \to a^+$ is a is a "left endpoint" of the domain. Similarly, $x \to a$ should be understood as $x \to b^-$ if b is a "right endpoint" of the domain.

- Sometimes, d is called a *point of discontinuity* of a function f if the condition
$$\lim_{x\to a} f(x) = f(d)$$
is not satisfied, i.e., either
  - $f(d)$ is not well-defined;
  - or or the limit does not exist at all;
  - or $f(d)$ is well-defined but not equal to the well-defined limit $\lim_{x\to a} f(x)$.

  According to this definition, every point not in the domain of f could be considered as a point of discontinuity of the function, which is sometime confusing.

Example

It is easy to see where the functions are continuous/discontinuous:



(i) Point of discontinuity: $x = 1, 3$, or 6. (Continuous at every point in the domain except $x = 1, 3$.)

(ii) Point of discontinuity: $x = 1, 2, 3$, or 4. (Continuous at every point in the domain except $x = 1, 2, 3$.)

(iii) Continuous at every point in the domain of the function.

(checking across which point the graph breaks into "separate pieces").

Continuity of Functions

Example

Find the value of the constant k such that the following piecewise polynomial function is continuous everywhere.

$$f(x) = \begin{cases} x^2 + 3x - 2k & \text{if } x \leq 1, \\ 2x - 3k & \text{if } x > 1. \end{cases}$$

It is easy to check that for any $a \neq 1$, $\lim\limits_{x \to a} f(x) = f(a)$.

We now check the continuity condition of f at 1. Note that

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x^2 + 3x - 2k) = 4 - 2k = f(1)$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x - 3k) = 2 - 3k$$

To make f continuous at $x = 1$, we need to pick k so that $4 - 2k = 2 - 3k$, i.e., $k = -2$.

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Properties of Continuous Functions

- Sums, differences, products of continuous functions are continuous.
- In particular, polynomial functions are continuous on the entire real line. Recall here that a polynomial function of degree n is a function of the form

$$f(x) = ax^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

where $a_0, a_1, \ldots, a_n$ are real numbers, and n is a non-negative integer.

Properties of Continuous Functions

- If two functions $f(x)$, $g(x)$ are continuous at $x = c$ and $g(c) \neq 0$, then the quotient $\frac{f}{g}$ is continuous at $x = c$.
- In particular, rational functions are continuous on the real line, except at the zeros of their denominators, i.e., continuous on their domains. Recall here that a *rational function* is a function of the form

  $$f(x) = \frac{p(x)}{q(x)},$$

  where $p(x), q(x)$ are polynomials with $q(x) \not\equiv 0$.
- For any positive integer n, the root function $f^{1/n}$ of a function f continuous at $x = c$ is also continuous at $x = c$, as long as the power function is well-defined on an open interval containing c.

Properties of Continuous Functions

These properties are straightforward consequences of the limit laws.

For example, if f and g are continuous at a, then

$$\lim_{x\to a} f(x) = f(a), \qquad \lim_{x\to a} g(x) = g(a)$$

and hence

$$\begin{aligned}
&\lim_{x\to a}(f+g)(x) \\
=&\lim_{x\to a}(f(x)+g(x)) \\
=&\lim_{x\to a} f(x) + \lim_{x\to a} g(x) \\
=&f(a)+g(a) = (f+g)(a)
\end{aligned}$$

i.e., the function $f+g$ is also continuous at a.

The (ε, δ)-Definition of Continuity

The (ε, δ)-Definition of Continuity

Given a function f whose domain is D and an element x_0 in D, f is said to be continuous at the point x_0 when the following holds:

For any real number $\varepsilon > 0$, there exists some number $\delta > 0$ such that for all x in the domain of f with

$$|x - x_0| < \delta,$$

the value of $f(x)$ satisfies

$$|f(x) - f(x_0)| \leq \varepsilon.$$

The elementary functions $\sin x$, $\cos x$, $\tan x$, a^x and $\log_a x$ are all continuous at any point in their domains. We can check their graphs or prove the continuity by (ε, δ)-definition.

Properties of Continuous Functions

- Note also that if f is continuous at c and g is continuous at $f(c)$, then the composition of the two functions $g \circ f$ is continuous at c.
- In fact, as $x \to c$, $f(x) \to f(c)$ by the continuity of f at c, and hence $g(f(x)) \to g(f(c))$ by the continuity of g at $f(c)$.

Exercise

By drawing graphs, find some examples of f and g so that $g \circ f$ is continuous at c, while

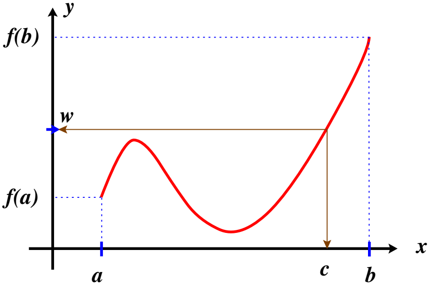
- g is not continuous at $f(c)$;
- or f is not continuous at c.

Outline

Intermediate Value Theorem

Theorem (Intermediate Value Theorem)

Suppose the function $y = f(x)$ is continuous on a closed interval $[a, b]$ and let w be a real number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there must be a number c in (a, b) such that $f(c) = w$.



In other words, the equation $f(x) = w$ must have at least one root in the interval (a, b). The Intermediate Value Theorem is very useful in locating roots of equations.

Intermediate Value Theorem

Example

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ in the interval $(1, 2)$.

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$, which is continuous on [1,2]. Then 0 is a number between $f(1)$ and $f(2)$:

$$-1 = f(1) < 0 < f(2) = 12.$$

By the Intermediate Value Theorem, there must be a number c in $(1, 2)$ such that $f(c) = 0$.

Similarly, $f(1.5) = 3.4 > 0$, hence the equation must have a root in the interval $(1, 1.5)$. We can also compute $f(1.25)$ to determine the root lies in $(1.1.25)$ or $(1.25, 1.5)$.

Continuing in this manner, one can end up with the "Bisection Method" for locating approximate roots of equations.

Intermediate Value Theorem

By the Intermediate Value Theorem, the problem of solving an inequality of the form $f(x) < 0$ for any continuous function f is essentially the same as solving $f(x) = 0$.

Once the zeros or undefined point of $f(x)$ are all located, it is just a matter of sign checking for $f(x)$ in various intervals in order to solve the inequality $f(x) < 0$ or $f(x) > 0$.

Intermediate Value Theorem

For example, the roots and undefined points of

$$\frac{(x+3)(2x-5)}{x+2} = 0$$

are $x = -3, \frac{5}{2}$ and -2 respectively, which divide the real line into four disjoint open intervals:

$$(-\infty, -3), \quad (-3, -2), \quad \left(-2, \frac{5}{2}\right), \quad \left(\frac{5}{2}, \infty\right) \ .$$

Note that $\frac{(x+3)(2x-5)}{x+2}$ cannot change sign in each of these intervals, since no other root is possible. By putting in some x values in these intervals, it is easy to see that

$$f(x) \begin{cases} < 0 & \text{if } x < -3 \text{ or } -2 < x < \dfrac{5}{2} \\ > 0 & \text{if } -3 < x < -2 \text{ or } x > \dfrac{5}{2} \end{cases}$$

Outline

Limit Definition of Derivatives

Recall that the rate of change of a function $y = f(x)$ at $x = a$ is a certain limit called the *derivative of f at a* , which is denoted by $f'(a)$, and is defined by

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \overset{or}{=} \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

whenever the limit exists.

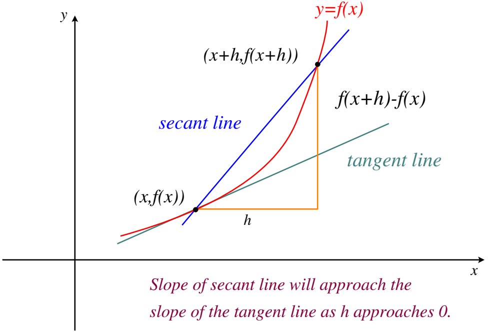
- The function f is said to be *differentiable at $x = a$* when $f'(a)$ exists on real numbers. (only correct for single variable calculus)
- Recall also that the limit $f'(a)$ can be interpreted as the slope of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$.

Limit Definition of Derivatives

If we want to measure how fast the function value $y = f(x)$ changes as x varies, we consider the *derivative function* $f'(x)$, which is defined as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

whenever the limit exists. Geometrically speaking, f' is the slope function of f.



Limit Definition of Derivatives

Some other often used notations to denote the derivative $f'(x)$ of the function $y = f(x)$ are as follows:

$$\frac{df}{dx}, \quad \frac{dy}{dx}, \quad y', \quad \text{and} \quad \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{dy}{dx}\right|_{x=a} = y'(a) = f'(a).$$

The process of finding the derivative of a given function is called differentiation.

When computing derivatives by using the limit definition of derivative, it is sometimes called differentiating by the first principle.

Examples of Derivatives

Example

Find the equation of the tangent line to the graph of the function $y = f(x) = 2x^2 - 3$ at the point $(1, -1)$.

The slope of the tangent line passing through $(1, -1)$ is

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0} \frac{[2(1+h)^2 - 3] - [2 \cdot 1^2 - 3]}{h}$$

$$= \lim_{h \to 0} \frac{2 + 4h + 2h^2 - 3 - 2 + 3}{h} = \lim_{h \to 0} (4 + 2h) = 4$$

Therefore the slope of the tangent line at $(1, -1)$ is 4, and the equation of the tangent line is given by

$$\frac{y - (-1)}{x - 1} = 4 \Longrightarrow y = 4x - 5$$

Examples of Derivatives

Example

Given function $y = f(x) = 2x^2 - 3$, find the derivative function $f'(x)$.

The derivative function $f'(x)$, by the limit definition of derivative (or the "first principle"), is given by the limit

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{[2(x+h)^2 - 3] - [2x^2 - 3]}{h} \\ &= \lim_{h \to 0} \frac{4xh + 2h^2}{h} \\ &= \lim_{h \to 0} (4x + 2h) \\ &= 4x \end{aligned}$$

Examples of Derivatives

Example

Differentiate the function

$$f(x) = \frac{1}{x+2}$$

by using the limit definition of derivative.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{(x+h)+2} - \frac{1}{x+2}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(x+2) - (x+h+2)}{(x+h+2)(x+2)}}{h} = \lim_{h\to 0} \frac{-1}{(x+h+2)(x+2)}$$

$$= -\frac{1}{(x+2)^2}$$

Examples of Derivatives

Example

Differentiate the function

$$g(x) = \sqrt{2x-1}$$

by using the limit definition of derivative.

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{2(x+h)-1}-\sqrt{2x-1}}{h} \\
&= \lim_{h\to 0} \frac{(\sqrt{2x+2h-1}-\sqrt{2x-1})(\sqrt{2x+2h-1}+\sqrt{2x-1})}{h(\sqrt{2x+2h-1}+\sqrt{2x-1})} \\
&= \lim_{h\to 0} \frac{2}{\sqrt{2x+2h-1}+\sqrt{2x-1}} \\
&= \frac{1}{\sqrt{2x-1}}
\end{aligned}$$

Differentiable and Continuous

Theorem

If f is differentiable at a point $x = a$, then f is continuous at $x = a$.

Proof.

We have

$$
\begin{aligned}
\lim_{x \to a} f(x) - f(a) = & \lim_{x \to a} (f(x) - f(a)) \\
= & \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right] \\
= & \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} (x - a) \\
= & f'(a) \cdot 0 = 0
\end{aligned}
$$

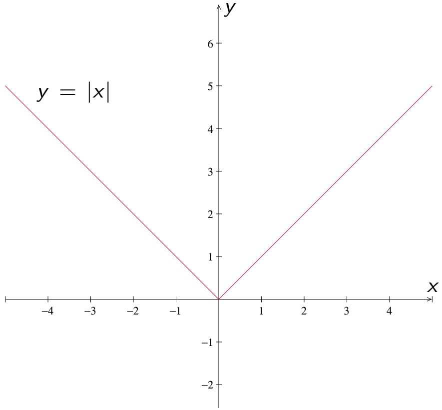
That is, $\lim_{x \to a} f(x) = f(a)$ and hence the function is continuous at a. □

Outline

Non-Differentiability

The derivative of a continuous function may not exist at every point.

A basic example is $f(x) = |x|$. Its derivative at $x = 0$, namely $f'(0)$, does not exist since there is no tangent line to the graph at $(0, 0)$.



Non-Differentiability

More precisely, by the limit definition of derivative, we have

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{|h| - 0}{h}$$

but

$$\lim_{h\to 0^+} \frac{|h| - 0}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1, \quad \lim_{h\to 0^-} \frac{|h| - 0}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

i.e., the one-sided limits do not agree and therefore the limit does not exist.

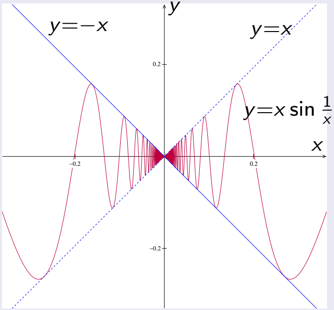
Exercise

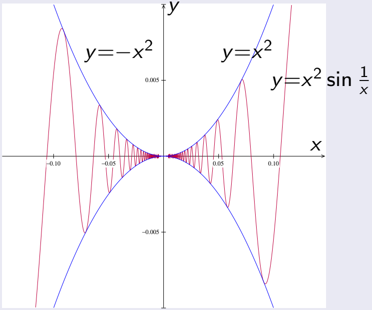
Show that $f(x) = |x|$ is differentiable at $x = x_0$ whenever $x_0 \neq 0$.

Exercise

Show by working the limit definition of derivative that $f'(0)$ and $g'(0)$ do not exist where

$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases} \quad \text{and} \quad g(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$





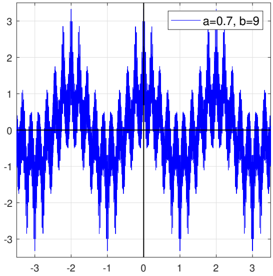
Weierstrass Function

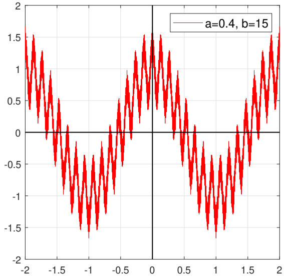
The Weierstrass function is an example of a real-valued function that is continuous everywhere but differentiable nowhere.

In Weierstrass's original paper, the function was defined as follows:

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

where $0 < a < 1$, b is a positive odd integer and $ab > 1 + \frac{3}{2}\pi$.





Calculus IB: Lecture 09

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Outline

Outline

1. Basic Formulas of Derivatives
2. Rules of Differentiation
3. The Chain Rule

Basic Formulas of Derivatives

Here are the derivatives of some elementary functions, which are the results of some limit computations.

1. $\dfrac{dc}{dx} = 0$, for any constant c
2. $\dfrac{dx^p}{dx} = px^{p-1}$, for any constant p
3. $\dfrac{de^x}{dx} = e^x, \qquad \dfrac{d\ln x}{dx} = \dfrac{1}{x}$
4. $\dfrac{d\sin x}{dx} = \cos x, \qquad \dfrac{d\cos x}{dx} = -\sin x$

Basic Formulas of Derivatives: $\frac{dx^p}{dx} = px^{p-1}$

In the case of p is a non-negative integer, we have

$$\begin{aligned}
\frac{dx^p}{dx} &= \lim_{h\to 0} \frac{(x+h)^p - x^p}{h} \\
&= \lim_{h\to 0} \frac{\left[(x+h)-x\right]\left[(x+h)^{p-1} + (x+h)^{n-2}x + \cdots + x^{p-1}\right]}{h} \\
&= \lim_{h\to 0} \frac{h\left[(x+h)^{p-1} + (x+h)^{n-2}x + \cdots + x^{p-1}\right]}{h} \\
&= \lim_{h\to 0} \left[(x+h)^{p-1} + (x+h)^{n-2}x + \cdots + x^{p-1}\right] \\
&= \lim_{h\to 0} \Big[\underbrace{x^{p-1} + x^{p-2}x + \cdots + x^{p-1}}_{p \text{ terms}}\Big] \\
&= px^{p-1}
\end{aligned}$$

What about p is not a non-negative integer?

Basic Formulas of Derivatives: $\frac{de^x}{dx} = e^x$

Recall that we have

$$\lim_{h\to 0} \frac{e^h - 1}{h} = 1.$$

By the limit definition of derivative,

$$\begin{aligned}\frac{de^x}{dx} &= \lim_{h\to 0} \frac{e^{x+h} - e^x}{h} \\ &= \lim_{h\to 0} e^x \cdot \frac{e^h - 1}{h} \\ &= e^x \lim_{h\to 0} \frac{e^h - 1}{h} \\ &= e^x\end{aligned}$$

Basic Formulas of Derivatives: $\frac{d \ln x}{dx} = \frac{1}{x}$

Recall that we have

$$\ln x = \log_e x$$

and

$$e = \lim_{h \to 0} (1 + h)^{\frac{1}{h}}.$$

Using the definition of derivative, we have

$$\begin{aligned}
\frac{d \ln x}{dx} &= \lim_{h \to 0} \frac{\ln(x + h) - \ln x}{h} \\
&= \lim_{h \to 0} \frac{1}{h} \ln \left(\frac{x + h}{x} \right) \\
&= \lim_{h \to 0} \ln \left(1 + \frac{h}{x} \right)^{\frac{1}{h}}.
\end{aligned}$$

Let $y = \dfrac{h}{x}$ and $h \to 0$ means $y \to 0$, then

$$\begin{aligned}
\lim_{h\to 0} \ln\left(1 + \frac{h}{x}\right)^{\frac{1}{h}} &= \lim_{y\to 0} \ln\left(1 + y\right)^{\frac{1}{xy}} \\
&= \lim_{y\to 0} \ln\left[\left(1 + y\right)^{\frac{1}{y}}\right]^{\frac{1}{x}} \\
&= \lim_{y\to 0} \frac{1}{x} \ln\left(1 + y\right)^{\frac{1}{y}} \\
&= \frac{1}{x} \lim_{y\to 0} \ln\left(1 + y\right)^{\frac{1}{y}} \\
&= \frac{1}{x} \ln\left(\lim_{y\to 0}\left(1 + y\right)^{\frac{1}{y}}\right)
\end{aligned}$$

The last step is based on the following theorem.

Theorem

Suppose function f satisfies $\lim_{y \to x_0} f(y) = u_0$ and function g is continuous at u_0, then the composition function $(g \circ f)(y) = g(f(y))$ holds that

$$\lim_{y \to y_0} (g \circ f)(y) = \lim_{u \to u_0} g(u) = g(u_0).$$

Let $g(u) = \ln u$ and $f(y) = (1+y)^{\frac{1}{y}}$, we have

$$\lim_{y \to 0} \ln (1+y)^{\frac{1}{y}} = \ln \left(\lim_{y \to 0} (1+y)^{\frac{1}{y}} \right) = \ln e = 1,$$

since $g(u) = \ln u$ is continuous on its domain. Hence, we have $\dfrac{d \ln x}{dx} = \dfrac{1}{x}$.

Note that above theorem requires the continuity of g, rather than f.

Consider the following examples:

$$g(u) = \begin{cases} 2 & u = 2 \\ 1 & u \neq 2 \end{cases} \qquad \text{and} \qquad f(y) = 2.$$

Let $u_0 = y_0 = 2$. Then we have

$$\lim_{y \to y_0} (g \circ f)(y) = \lim_{y \to 2} (g \circ f)(y) = \lim_{y \to 2} g(f(y)) = \lim_{y \to 2} g(2) = 2.$$

On the other hand, we also have

$$u_0 = \lim_{y \to 2} f(y) = 2 \quad \text{and} \quad \lim_{u \to u_0} g(u) = \lim_{u \to 2} g(u) = 1 \neq \lim_{y \to y_0} (g \circ f)(y).$$

Outline

1. Basic Formulas of Derivatives
2. Rules of Differentiation
3. The Chain Rule

Rules of Differentiation

Whenever f' and g' both exist, we have the following rules:

1. $\dfrac{d}{dx}(af+bg) = a\dfrac{df}{dx} + b\dfrac{dg}{dx} = af' + bg'$ for any constants a and b.
2. **Product Rule:** $\dfrac{d}{dx}(fg) = f\dfrac{dg}{dx} + g\dfrac{df}{dx} = fg' + gf'$
3. **Quotient Rule:** $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} = \dfrac{gf' - fg'}{g^2}$

Exercise

Prove the first rule $\dfrac{d}{dx}(af+bg) = a\dfrac{df}{dx} + b\dfrac{dg}{dx} = af' + bg'$.

The Proof of Product Rule

We prove this rule by the limit definition of derivative:

$$
\begin{aligned}
&(fg)'(x) \\
=&\lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
=&\lim_{h\to 0}\frac{(f(x+h)g(x+h)-f(x+h)g(x))+(f(x+h)g(x)-f(x)g(x))}{h} \\
=&\lim_{h\to 0}\frac{f(x+h)(g(x+h)-g(x))}{h}+\lim_{h\to 0}\frac{g(x)(f(x+h)-f(x))}{h} \\
=&\lim_{h\to 0}f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
=&f(x)g'(x)+g(x)f'(x)
\end{aligned}
$$

where we use $\lim_{h\to 0} f(x+h) = f(x)$ since a function is continuous at any point where f' exists.

Exercise

Prove the quotient rule: $\dfrac{d}{dx}\left(\dfrac{f}{g}\right) = \dfrac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2} = \dfrac{gf' - fg'}{g^2}.$

Outline

The Chain Rule

Let F is compositions of two functions f and g:

$$F(x) = (f \circ g)(x) = f(g(x)),$$

such that

1. g is a differentiable at x (the derivative $g'(x)$ exists),
2. and f is a is differentiable at $g(x)$ (the derivative $f'(g(x))$ exists);

then $y = F(x) = (f \circ g)(x)$ is differentiable at x, and its derivative is

$$F'(x) = f'(g(x)) \cdot g'(x).$$

The Proof of Chain Rule

There is one idea for the proof

$$F'(x) = \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

$$= \lim_{h\to 0} \left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}\right]$$

$$= \underbrace{\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}}_{\text{slope of } f \text{ at } g(x)} \cdot \underbrace{\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}}_{\text{slope of } g(x) \text{ at } x}$$

$$= f'(g(x)) \cdot g'(x)$$

How to proof this equality?

The remain is to show that

$$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} = f'(g(x)).$$

The Proof of Chain Rule

Recall the (ε, δ)-definition of limit.

Definition

The expression $\lim_{h \to 0} w(x) = L$ means for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|w(h) - L| < \varepsilon$ whenever $0 < |h| < \delta$.

We must find $\delta > 0$ such that $w(h)$ is well defined whenever

$$0 < |h| < \delta.$$

However, the function

$$w(h) = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}$$

is not well defined when $g(x+h) - g(x) = 0$.

The Proof of Chain Rule

For constant function such that $g(x) = a$, it is obviously

$$w(h) = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}$$

is undefined.

In such case, we have

$$F(x) = f(g(x)) = f(a),$$

which is also a constant function, so that $F'(x) = 0$.

If $g(x)$ is any differentiable but not constant function, is it true that we can always find a constant $\delta > 0$ such that $w(h)$ is well defined whenever h in $(-\delta, 0) \cup (0, \delta)$ (that is $0 < |h| < \delta$)?

The Proof of Chain Rule

Unfortunately, this guess is also INCORRECT!

There exists function which is differentiable at 0 but $w(h)$ is not well defined in $(-\delta, 0) \cup (0, \delta)$ for any $\delta > 0$.

Consider the function

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

In the case of $x = 0$, using squeeze theorem, we have

$$g'(x) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = \lim_{h \to 0} \frac{h^2 \sin \frac{1}{h}}{h} = \lim_{h \to 0} h \sin \frac{1}{h} = 0.$$

Hence, g is differential at 0, which satisfies the condition of chain rule.

The Proof of Chain Rule

For $x = 0$, the function

$$w(h) = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}$$

is undefined means $g(h) - g(0) = 0$, that is $h^2 \sin \frac{1}{h} = 0$.

For $h = \frac{1}{n\pi}$ with any integer n, we have

$$h^2 \sin \frac{1}{h} = \frac{1}{(n\pi)^2} \sin(n\pi) = 0.$$

Hence, for any $\delta > 0$, we can take

$$n_0 = \left\lceil \frac{1}{\delta\pi} \right\rceil \quad \text{and} \quad h = \frac{1}{n_0\pi}$$

Then we have $h^2 \sin \frac{1}{h} = 0$ and $0 < h < \delta$.

The Proof of Chain Rule

The idea on Page 18 does NOT work!

$$
\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h\to 0} \left[\frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \right] \\
&= \underbrace{\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}}_{\text{This term may be undefined!}} \cdot \underbrace{\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}}_{\text{slope of } g(x) \text{ at } x} \\
&= f'(g(x)) \cdot g'(x) \qquad \text{This step is WRONG!!!}
\end{aligned}
$$

SOS!!! How to Prove the Chain Rule???

$$\underbrace{\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}}_{\text{This term may be undefined!}} \cdot \underbrace{\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}}_{\text{slope of } g(x) \text{ at } x}$$

$$= f'(g(x)) \cdot g'(x)$$

This step is WRONG!!!

SOS!!! How to Prove the Chain Rule??????

We can replace the term

$$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)}$$

by something which is well-defined!

The Proof of Chain Rule

We introduce a function as follows:

$$Q(y) = \begin{cases} \dfrac{f(y) - f(g(x))}{y - g(x)}, & y \neq g(x), \\ f'(g(x)), & y = g(x). \end{cases}$$

We can show

$$Q(g(x+h)) \cdot \frac{g(x+h) - g(x)}{h} \quad \text{is equal to} \quad \frac{f(g(x+h)) - f(g(x))}{h}.$$

1. Whenever $g(x+h)$ is not equal to $g(x)$, we have

$$\begin{aligned} & Q(g(x+h)) \cdot \frac{g(x+h) - g(x)}{h} \\ = & \frac{f(y) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} = \frac{f(g(x+h)) - f(g(x))}{h} \end{aligned}$$

2. When $g(x+h)$ equals $g(x)$, both of them are zero.

The Proof of Chain Rule

We have

$$\begin{aligned}F'(x) &= \lim_{h\to0}\frac{f(g(x+h))-f(g(x))}{h}\\ &= \lim_{h\to0}\left(Q(g(x+h))\cdot\frac{g(x+h)-g(x)}{h}\right)\\ &= \lim_{h\to0}Q(g(x+h))\cdot\underbrace{\lim_{h\to0}\frac{g(x+h)-g(x)}{h}}_{g'(x)}\end{aligned}$$

Since both g and Q are continuous, we have

$$\lim_{h\to0}Q(g(x+h)) = Q\left(\lim_{h\to0}g(x+h)\right) = Q\left(g(x)\right) = f'(g(x)).$$

The Proof of Chain Rule

Let $u = g(x)$, we can verify the continuity of Q at $g(x)$:

$$\begin{aligned}
\lim_{y \to g(x)} Q(y) &= \lim_{y \to u} Q(y) \\
&= \lim_{y \to u} \frac{f(y) - f(g(x))}{y - g(x)} \\
&= \lim_{y \to u} \frac{f(y) - f(u)}{y - u} \\
&= f'(u) = f'(g(x)) = Q(g(x)).
\end{aligned}$$

Combing all above results, we can prove the chain rule

$$F'(x) = \lim_{h \to 0} Q(g(x+h)) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f'(g(x)) \cdot g'(x).$$

Examples of Chain Rule

Now, we can show $\dfrac{dx^p}{dx} = px^{p-1}$ for any constant exponent p.

Proof.

The definition of ln means $x^p = e^{\ln x^p} = e^{p \ln x}$, then

$$
\begin{aligned}
\frac{dx^p}{dx} &= \frac{d\left(e^{p\ln x}\right)}{dx} \\
&= \frac{de^u}{du} \cdot \frac{d\left(p \ln x\right)}{dx} && \text{Chain rule with } f(u) = e^u \text{ and } g(x) = p \ln x \\
&= e^u \cdot p \cdot \frac{d \ln x}{dx} && \text{Using } \frac{de^u}{du} = e^u \text{ and } \frac{d \ln x}{x} = \frac{1}{x} \\
&= x^p \cdot p \cdot \frac{1}{x} && \text{Using } \frac{d \ln x}{x} = \frac{1}{x} \\
&= px^{p-1}
\end{aligned}
$$

$\square$

Calculus IB: Lecture 10

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Outline

Outline

1. Derivatives of Trigonometric Functions

2. Derivatives of Inverse Functions

3. Implicit Differentiation

Derivatives of Trigonometric Functions

1. $$\frac{d \sin x}{dx} = \cos x$$
2. $$\frac{d \cos x}{dx} = -\sin x$$
3. $$\frac{d \tan x}{dx} = \sec^2 x$$
4. $$\frac{d \cot x}{dx} = -\csc^2 x$$
5. $$\frac{d \sec x}{dx} = \sec x \tan x$$
6. $$\frac{d \csc x}{dx} = -\csc x \cot x$$

Derivatives of $\sin x$

Using the identities (Lecture-L04)

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

with $\alpha = x + h$ and $\beta = x$, we have

$$\begin{aligned}
\frac{d\sin x}{dx} &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{2\cos\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} \\
&= \lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\cdot\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \\
&= \cos\left(\lim_{h\to 0}\left(x+\frac{h}{2}\right)\right)\cdot\lim_{t\to 0}\frac{\sin t}{t} \qquad \text{let } t=\frac{h}{2} \\
&= \cos x\cdot 1 = \cos x.
\end{aligned}$$

Derivatives of $\cos x$

Using the identities (Lecture-L04)

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

with $\alpha = x+h$ and $\beta = x$, we have

$$\begin{aligned}
\frac{d\cos x}{dx} &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \\
&= \lim_{h\to 0}\frac{-2\sin\left(x+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{h} \\
&= -\lim_{h\to 0}\sin\left(x+\frac{h}{2}\right)\cdot\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \\
&= -\sin\left(\lim_{h\to 0}\left(x+\frac{h}{2}\right)\right)\cdot\lim_{t\to 0}\frac{\sin t}{t} \qquad \text{let } t=\frac{h}{2} \\
&= -\sin x\cdot 1 = -\sin x.
\end{aligned}$$

Derivatives of $\tan x$

By using $\dfrac{d\sin x}{dx} = \cos x$, $\dfrac{d\cos x}{dx} = -\sin x$ and quotient rule we have

$$\begin{aligned}
\frac{d\tan x}{dx} &= \frac{d}{dx}\frac{\sin x}{\cos x} \\
&= \frac{\cos x \dfrac{d\sin x}{dx} - \sin x \dfrac{d\cos x}{dx}}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x.
\end{aligned}$$

More Examples

Example

Find the derivative of $\dfrac{e^x \sin x}{x^2}$.

$$\frac{d}{dx}\frac{e^x \sin x}{x^2} = \frac{x^2 \dfrac{de^x \sin x}{dx} - e^x \sin x \dfrac{dx^2}{dx}}{x^4} \qquad \text{(Quotient Rule)}$$

$$= \frac{x^2 \left(e^x \dfrac{d \sin x}{dx} + \sin x \dfrac{de^x}{dx} \right) - 2xe^x \sin x}{x^4} \qquad \text{(Product Rule)}$$

$$= \frac{x^2 (e^x \cos x + e^x \sin x) - 2xe^x \sin x}{x^4}$$

$$= \frac{e^x (x \cos x + x \sin x - 2 \sin x)}{x^3}$$

Exercise of Derivatives

Exercise

Show that

1. $$\frac{d\cot x}{dx} = -\csc^2 x,$$
2. $$\frac{d\sec x}{dx} = \sec x \tan x,$$
3. $$\frac{d\csc x}{dx} = -\csc x \cot x.$$

More Examples

Example

Differentiate $y = (3x^4 - 2x^2 + 1)^3$.

Let $u = 3x^4 - 2x^2 + 1$, then $y = u^3$, $\dfrac{dy}{du} = 3u^2$, and

$$\frac{du}{dx} = 3 \cdot 4x^{4-1} - 2 \cdot 2x^{2-1} + 0 = 12x^3 - 4x.$$

Hence by the chain rule,

$$\begin{aligned}\frac{dy}{dx} =& \frac{dy}{du} \cdot \frac{du}{dx} \\ =& 3u^2 \cdot (12x^3 - 4x) \\ =& 3(3x^4 - 2x^2 + 1)^2(12x^4 - 4x).\end{aligned}$$

More Examples

Example

Differentiate $y = \left(\frac{x}{1+x}\right)^{\frac{1}{3}}$.

Let $u = \frac{x}{1+x}$, then $y = \sqrt[3]{u} = u^{\frac{1}{3}}$. We have

$$\frac{dy}{du} = \frac{1}{3}u^{\frac{1}{3}-1} = \frac{1}{3}u^{-\frac{2}{3}},$$

$$\frac{du}{dx} = \frac{(1+x)\cdot\frac{dx}{dx} - x\frac{d(1+x)}{dx}}{(1+x)^2} = \frac{1}{(1+x)^2}.$$

By the chain rule, we obtain

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{3}\left(\frac{x}{1+x}\right)^{-\frac{2}{3}}\frac{1}{(1+x)^2} = \frac{1}{3}x^{-\frac{2}{3}}(1+x)^{-\frac{4}{3}}$$

More Examples

Exercise

Define $y = (1 + x^2)\sin(2x^2 + e^{x^2}) + e^{\sin \ln x}$. Show that

$$\begin{aligned}\frac{dy}{dx} =& 2x\sin(2x^2 + e^{x^2}) + (1 + x^2)\cos(2x^2 + e^{x^2})(4x + 2xe^{x^2}) \\ &+ \frac{1}{x} e^{\sin \ln x} \cos \ln x.\end{aligned}$$

Outline

Derivatives of Inverse Functions

Theorem (Derivatives of Inverse Function)

Suppose f is a differentiable and has inverse function f^{-1} over an interval I and x is a point in I such that $x = f(a)$ and $f'(a) \neq 0$, then f^{-1} is differentiable at x and its derivative is

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(x))}.$$

Proof.

The definition of inverse function means $f(f^{-1}(x)) = x$. We can regard $f(f^{-1}(x))$ as composition of f and f^{-1} and use chain rule to obtain

$$\left(f \circ f^{-1}\right)'(x) = 1 \Longrightarrow f'(f^{-1}(x)) \cdot \left(f^{-1}\right)'(x) = 1 \Longrightarrow \left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}$$

□

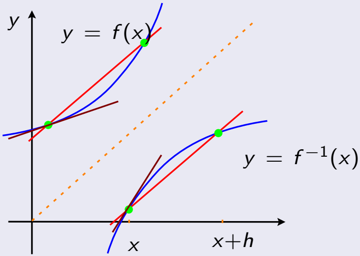
Derivatives of Inverse Functions

Exercise

By reflection across the line $y = x$, the tangent line to the graph of $y = f^{-1}(x)$ at the point $(x, f^{-1}(x))$ is reflected to the tangent line to the graph of $y = f(x)$ at $(f^{-1}(x), x)$. Try to explain

$$\left(f^{-1}\right)'(x) = \frac{1}{f'(f^{-1}(x))}$$

in geometric view.



Example

Let $y = \ln x = f^{-1}(x)$ where $f(x) = e^x$.

Then $f'(x) = e^x$, and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

which means $\dfrac{d\ln x}{dx} = \dfrac{1}{x}$ we have proved before.

Derivatives of Inverse Functions

Example

If h is the inverse function of the increasing function $f(x) = x^3 + x + 1$, find $h'(1)$.

Note that we have

$$f'(x) = 3x^2 + 1.$$

Moreover, it is easy to verify $f(0) = 1$ which means $h(1) = 0$.

Hence, we have

$$h'(1) = \frac{1}{f'(h(1))} = \frac{1}{f'(0)} = \frac{1}{3 \cdot 0 + 1} = 1.$$

Derivatives of Inverse Functions

Example

Find the derivative of $\tan^{-1} x$.

Define $f(x) = \tan x$ with domain $-\pi/2 < x < \pi/2$. Then we have

$$f^{-1}(x) = \tan^{-1} x \quad \text{and} \quad f'(x) = \sec^2 x.$$

Hence, we have

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sec^2(\tan^{-1} x)} = \frac{1}{1 + \tan^2(\tan^{-1}(x))} = \frac{1}{1 + x^2},$$

where we use the identity (consider that $u = \tan^{-1} x$)

$$\frac{1}{\sec^2 u} = \frac{\cos^2 u}{\sin^2 u + \cos^2 u} = \frac{1}{\frac{\sin^2 u}{\cos^2 u} + 1} = \frac{1}{\tan^2 u + 1}$$

In other words, we have $\dfrac{d \tan^{-1} x}{dx} = \dfrac{1}{1 + x^2}$.

Outline

Implicit Differentiation

Sometimes, a function $y = f(x)$ can be defined implicitly by an equation of x and y of the form

$$F(x, y) = 0.$$

For example, the unit circle can be defined as

$$x^2 + y^2 = 1.$$

By solving the equation, we obtain two functions

$$y = \sqrt{1 - x^2} \quad \text{with} \quad y' = \frac{dy}{dx} = -\frac{x}{\sqrt{1 - x^2}}$$

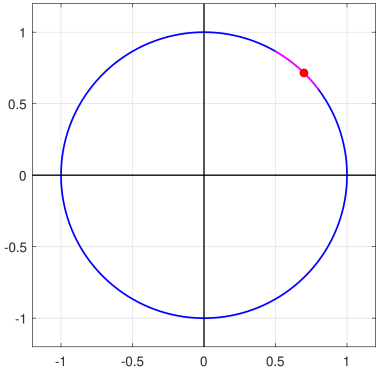
and

$$y = -\sqrt{1 - x^2} \quad \text{with} \quad y' = \frac{dy}{dx} = \frac{x}{\sqrt{1 - x^2}}.$$

The Implicit Function Theorem (beyond the requirement)

Graphically, we can observe $x^2 + y^2 = 1$ fails the familiar vertical line test, but it could pass the vertical line test locally.

Most of points on the circle we can choose a small neighborhood where our curve satisfies the vertical line test (determines y as a function of x).



The Implicit Function Theorem (beyond the requirement)

Graphically, we can observe $x^2 + y^2 = 1$ fails the familiar vertical line test, but it could pass the vertical line test locally.

Theorem (The Implicit Function Theorem)

Consider a continuously differentiable function $F(x, y)$ *and a point* (x_0, y_0) *so that* $F(x_0, y_0) = c$*. If*

$$\frac{\partial F}{\partial y}(x_0, y_0) = 0,$$

then there is a neighborhood of (x_0, y_0) *so that whenever* x *is sufficiently close to* x_0 *there is a unique* y *so that* $F(x, y) = c$*. Moreover, this assignment is makes* y *a continuous function of* x*.*

Implicit Differentiation

In general, it is difficult or impossible to find the explicit expression of $y = f(x)$ by $F(x, y)$, but we can express $y' = f'(x)$ by x and y.

We desire to find $f'(x)$ directly from the implicit form $F(x, y) = 0$ without solving $y = f(x)$.

Implicit differentiation can be done as follows:

$$F(x, y) = 0 \quad \xrightarrow{\frac{d}{dx}\text{ both sides}} \quad \text{an equation to solve for } \frac{dy}{dx}$$

Implicit Differentiation

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Implicit differentiation can be done as follows:

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Example

Find the derivative of function $y = f(x)$ from the equation of unit circle.

We take the differentiation on both sides of the equation:

$$
\begin{aligned}
x^2 + y^2 = 1 \Longrightarrow & \frac{dx^2}{dx} + \frac{dy^2}{dx} = \frac{d1}{dx} \\
\Longrightarrow & 2x + \frac{dy^2}{dy} \cdot \frac{dy}{dx} = 0 \\
\Longrightarrow & 2x + 2y \cdot y' = 0 \\
\Longrightarrow & y' = -\frac{x}{y}.
\end{aligned}
$$

We can obtain a unique expression for the slope of the tangent line of unit circle at point (x, y) without the expression of $y = f(x)$.

Implicit Differentiation

Example

Find the derivative of function $y = f(x)$ from equation $\sin(xy) = x^2 + y$.

We take the differentiation on both sides of the equation:

$$\begin{aligned}
\sin(xy) = x^2 + y \Longrightarrow & \frac{d\sin(xy)}{dx} = \frac{dx^2}{dx} + \frac{dy}{dx} \\
\Longrightarrow & \cos(xy) \cdot \frac{dxy}{dx} = 2x + \frac{dy}{dx} \\
\Longrightarrow & \cos(xy) \cdot \left(x \cdot \frac{dy}{dx} + \frac{dx}{dx} \cdot y\right) = 2x + \frac{dy}{dx} \\
\Longrightarrow & y' = \frac{dy}{dx} = \frac{y\cos(xy) - 2x}{1 - x\cos(xy)}.
\end{aligned}$$

It is impossible to find explicit expression for $y = f(x)$ for this example.

Implicit Differentiation

Example

Find the derivative of $y = \sqrt[3]{2 - 2x^2}$.

The function can defined by equation $2x^2 + y^3 = 2$.

Just differentiate both sides as functions of x to get

$$\frac{d2x^2}{dx} + \frac{dy^3}{dx} = \frac{d2}{dx} \Longrightarrow 4x + 3y^2\frac{dy}{dx} = 0,$$

Then we have

$$\frac{dy}{dx} = -\frac{4x}{3y^2} = -\frac{4}{3}x(2 - 2x^2)^{-\frac{2}{3}}.$$

Example

Find derivatives of $y = \sin^{-1} x$.

We have $x = \sin y$ with $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and

$$
\begin{aligned}
x = \sin y \Longrightarrow & \frac{dx}{dx} = \frac{d \sin y}{dx} \\
\Longrightarrow & 1 = \frac{d \sin y}{dy} \cdot \frac{dy}{dx} \\
\Longrightarrow & 1 = \cos y \cdot \frac{dy}{dx}
\end{aligned}
$$

Since $\sin^2 y + \cos^2 y = 1$ and $\cos y \geq 0$ whenever $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, we have

$$
\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}.
$$

Exercise

Find derivatives of

$$\cos^{-1} x, \quad \cot^{-1} x, \quad \csc^{-1} x \quad \text{and} \quad \sec^{-1} x.$$

by implicit differentiation.

Chain Rule Version of Basic Derivative Formulas

The following chain rule versions of basic derivative formulas are convenient to use for calculation of derivatives.

$$\frac{d\blacksquare^p}{dx} = p\blacksquare^{p-1} \cdot \frac{d\blacksquare}{dx} \qquad \frac{de^{\blacksquare}}{dx} = e^{\blacksquare} \cdot \frac{d\blacksquare}{dx}$$

$$\frac{d\ln\blacksquare}{dx} = \frac{1}{\blacksquare} \cdot \frac{d\blacksquare}{dx} \qquad \frac{d\sin\blacksquare}{dx} = \cos\blacksquare \cdot \frac{d\blacksquare}{dx}$$

$$\frac{d\cos\blacksquare}{dx} = -\sin\blacksquare \cdot \frac{d\blacksquare}{dx} \qquad \frac{d\tan\blacksquare}{dx} = \sec^2\blacksquare \cdot \frac{d\blacksquare}{dx}$$

$$\frac{d\sec\blacksquare}{dx} = \sec\blacksquare \cdot \tan\blacksquare \cdot \frac{d\blacksquare}{dx} \qquad \frac{d\sin^{-1}\blacksquare}{dx} = \frac{1}{\sqrt{1-\blacksquare^2}} \cdot \frac{d\blacksquare}{dx}$$

$$\frac{d\tan^{-1}\blacksquare}{dx} = \frac{1}{1+\blacksquare^2} \cdot \frac{d\blacksquare}{dx}$$

Logarithmic Differentiation

Logarithmic differentiation is just a special case of implicit differentiation.

Example

Find the derivative of $y = \sqrt[3]{2 - 2x^2}$ by working with the equation

$$\ln y = \ln(2 - 2x^2)^{1/3} = \frac{1}{3}\ln(2 - 2x^2).$$

Just differentiating both sides as functions of x again:

$$\frac{d}{dx}\ln y = \frac{1}{3}\frac{d}{dx}\ln(2 - 2x^2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3} \cdot \frac{1}{2 - 2x^2} \cdot (-4x)$$

$$\frac{dy}{dx} = \frac{y}{3} \cdot \frac{-4x}{2 - 2x^2} = -\frac{4}{3}x(2 - 2x^2)^{-\frac{2}{3}}$$

Calculus IB: Lecture 11

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Outline

Outline

Rates of Change

When a function $y = f(x)$ describes the relation between two quantities represented by x and y respectively, the derivative function

$$f'(x) \quad \text{or} \quad \frac{dy}{dx}$$

is considers as the *rate of change* of the quantity y with respect to the quantity x.

Related Rates

The main idea about related rates is essentially the following.

Given some quantities

$$\left.\begin{array}{l} q_1 = q_1(t) \\ q_2 = q_2(t) \\ \vdots \\ q_n = q_n(t) \end{array}\right\} \qquad \begin{array}{c} \text{which are all functions of } t, \\ \text{where } t \text{ may represent time or some other quantity,} \end{array}$$

if there is an equation relating all these quantities, then

$$\frac{d}{dt} \text{ of both sides of the relation}$$

$$\xrightarrow{gives} \quad \text{an equation relating the rates of changes } \frac{dq_1}{dt}, \frac{dq_2}{dt}, \ldots, \frac{dq_n}{dt}$$

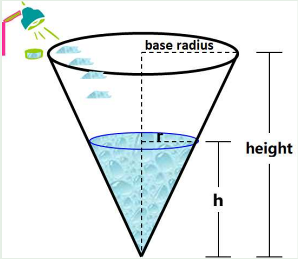
Examples of Related Rates

Example (cone container)

Water is flowing into a cone container at a rate of $2\mathrm{m}^3/\mathrm{min}$. Suppose the shape of the cone container satisfies

$$\frac{\text{base radius}}{\text{height}} = \frac{r}{h} = \frac{1}{2}.$$

How fast is the water level rising (speed of h increasing) when the water in the container is 3m in depth?



Example (cone container)

At time t, volume of water with depth h is $V = \frac{1}{3}\pi r^2 h$ and $r = \frac{h}{2}$.

We also know that $\frac{dV}{dt} = 2\mathrm{m}^3/\mathrm{min}$. The question is find $\left.\frac{dh}{dt}\right|_{h=3}$.

Here the equation relating V and h, both are functions of t, is

$$V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3$$

Taking the derivatives of both sides with respect to t,the chain rule means

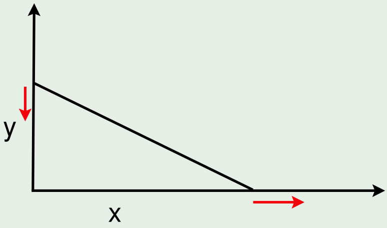
$$\frac{dV}{dt} = \frac{\pi}{4}h^2 \cdot \frac{dh}{dt}$$

When $h = 3$m, we have

$$2 = \frac{\pi}{4}(3)^2 \cdot \left.\frac{dh}{dt}\right|_{h=3} \quad \Longrightarrow \quad \left.\frac{dh}{dt}\right|_{h=3} = \frac{8}{9\pi}\ \mathrm{m}/\mathrm{min}$$

Example (ladder and wall)

A ladder of 5m long is leaning again the wall. If the bottom of the ladder is pulled away from the wall at a rate of 1m/min, how fast is the top of the ladder dropping when the bottom of the ladder is 3m away from the wall?



Example (ladder and wall)

At time t, we have:

- distance between the wall and the bottom of the ladder: $x = x(t)$
- distance between the ground and the top of the ladder: $y = y(t)$
- known condition: $\dfrac{dx}{dt} = 1\mathrm{m/min}$

We want to find $\left.\dfrac{dy}{dt}\right|_{x=3}$ and the relation between x and y is $x^2 + y^2 = 5^2$.

Differentiating both sides with respect to t, we have

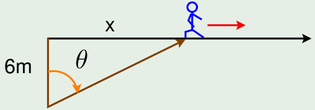
$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = \frac{d}{dt} \cdot (25) = 0.$$

When $x = 3$m, $y = \sqrt{5^2 - 3^2} = 4$m, and hence

$$2 \cdot 3 \cdot 1 + 2 \cdot 4 \cdot \left.\frac{dy}{dt}\right|_{x=3} = 0 \quad \Longrightarrow \quad \left.\frac{dy}{dt}\right|_{x=3} = -\frac{3}{4} \quad (\mathrm{m/min})$$

Example (walking man)

A man starts walking along a straight line at a velocity of 1.5m/s. A light beam 6m from the road is tracking the man. How fast is the angle of the beam turning when the man is 9m away from the starting point?



At time t, we have:

- distance travelled by the man : $x = x(t)$
- turning angle of the light beam: $\theta = \theta(t)$
- known condition: $\dfrac{dx}{dt} = 1.5 \ \mathrm{m/s}$

We want to find $\left.\dfrac{d\theta}{dt}\right|_{x=9}$.

Example (walking man)

The relation between x and θ is: $\tan\theta = \dfrac{x}{6}$.

Differentiating both sides with respect to t, we have

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{6}\frac{dx}{dt}$$

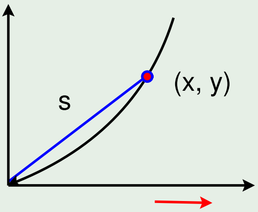
When $x = 9$m, $\sec\theta = \frac{\sqrt{9^2+6^2}}{6} = \frac{\sqrt{117}}{6}$m, and hence

$$\frac{117}{36} \cdot \left.\frac{d\theta}{dt}\right|_{x=9} = \frac{1}{6} \cdot 1.5 \quad \Longrightarrow \quad \left.\frac{d\theta}{dt}\right|_{x=9} = \frac{9}{117}\ (\mathrm{rad/s}).$$

Examples of Related Rates

Example (travelling along the curve)

A particle is travelling along the graph of the function $y = x^2$ such that the velocity of the particle in x-direction is 1m/s. How fast is the distance between the particle and the origin increasing when $x = 2$m?



At time t, we have:

- x coordinate of the particle: $x = x(t)$
- y coordinate of the particle: $y = y(t)$
- distance between the particle and the origin: $s = s(t)$
- known condition: $\dfrac{dx}{dt} = 1 \text{ m/s}$

Example (travelling along the curve)

We want to find $\left.\dfrac{ds}{dt}\right|_{x=2}$. Relations of the quantities:

$$\begin{cases} y = x^2 \\ s^2 = x^2 + y^2 \end{cases} \quad \xrightarrow{d/dt} \quad \begin{cases} \dfrac{dy}{dt} = 2x\dfrac{dx}{dt} \\ 2s\dfrac{ds}{dt} = 2x\dfrac{dx}{dt} + 2y\dfrac{dy}{dt} \end{cases}$$

When $x = 2\text{m}$, $y = 4\text{m}$, $s = \sqrt{2^2 + 4^2} = \sqrt{20}\text{m}$ and

$$\begin{cases} \left.\dfrac{dy}{dt}\right|_{x=2} = 2 \cdot 2 \cdot 1 = 4 \\ 2\sqrt{20} \cdot \left.\dfrac{ds}{dt}\right|_{x=2} = 2 \cdot 2 \cdot 1 + 2 \cdot 4 \cdot \left.\dfrac{dy}{dt}\right|_{x=2} = 4 + 8 \cdot 4 \end{cases}$$

Hence $\left.\dfrac{ds}{dt}\right|_{x=2} = \dfrac{18}{\sqrt{20}}$ m/s.

Outline

Second Order Derivative (or Second Derivative)

If $s = s(t)$ is the position function of a particle moving along a line represented by the x axis, then

$$\frac{dx}{dt} = \text{ velocity function } = v(t)$$

$$\frac{dv}{dt} = \text{ acceleration function} = a(t)$$

In particular, if m is the mass of the particle, and F is the force acting on the particle, Newton's Second Law $F = ma$ can be expressed as

$$F = m\frac{dv}{dt} = m\frac{d^2s}{dt^2}$$

where the second derivative means "the derivative of the derivative":

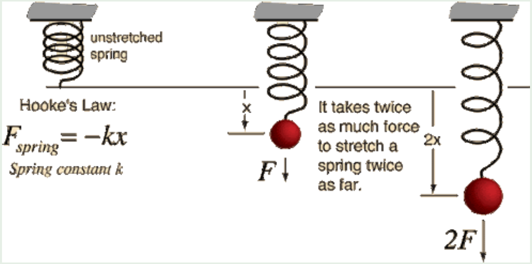
$$s''(t) = \frac{d^2s}{dt^2} \overset{means}{=} \frac{d}{dt}\left(\frac{ds}{dt}\right).$$

Example

The Hooke's Law for spring-mass system, which says that the spring force acting on a mass attached to the spring is proportional to the displacement of the mass from its equilibrium position, can be expressed as

$$m\frac{d^2s}{dt^2} = -kx$$

where k is the spring constant, and x is the displacement of the mass from its equilibrium position (spring is unstretched).



Higher Order Derivatives

The second order derivative of f is the derivative of the derivative of f:

$$\frac{d^2 f(x)}{dx^2} = f''(x) = (f')'(x).$$

The third order derivative of f is the derivative of the second order derivative of f:

$$\frac{d^3 f(x)}{dx^3} = f'''(x) = (f'')'(x).$$

In general, the n-th order derivative of f is the derivative of the $(n-1)$-th order derivative of f:

$$\frac{d^n f(x)}{dx^n} = f^{(n)}(x) = \left(f^{(n-1)}\right)'(x).$$

Higher Order Derivatives

Example

Calculate the second order derivative of $f(x) = \sin x^2$.

We apply the chain rule to calculate the first derivative:

$$f'(x) = (\cos x^2) \cdot 2x = 2x \cos x^2.$$

Then we need to use product rule and the chain rule to calculate the second derivative:

$$\begin{aligned} f''(x) &= \left(f'(x)\right)' \\ &= \left(2x \cos x^2\right)' \\ &=(2x)' \cdot \cos x^2 + 2x \cdot \left(\cos x^2\right)' \\ &=2 \cos x^2 - 4x^2 \sin x^2. \end{aligned}$$

Higher Order Derivatives

Example

For any polynomial $P_n(x)$ of degree n, show that $P_n^{(k)}(x) = 0$ for any integer $k > n$.

Since $\frac{d}{dx}(x^m)' = mx^{m-1}$, we have $\frac{d^k}{dx^k}(x^m) = 0$ if $k > m$.

Hence, if $k > n$, we have

$$\begin{aligned}
&\frac{d^k}{dx^k} P_n(x) \\
=&\frac{d^k}{dx^k}\left(a_0 + a_1 x + \cdots + a_n x^n\right) \\
=&a_0 \cdot \frac{d^k}{dx^k}(1) + a_1 \cdot \frac{d^k}{dx^k}(x) + \cdots + a_n \cdot \frac{d^k}{dx^k}(x^n) \\
=&0 + 0 + \cdots + 0 = 0.
\end{aligned}$$

Exercise

For any polynomial $P_n(x)$ of degree n and positive integer $k \leq n$, find $P_n^{(k)}(x)$.

Exercise

Let the function $y = f(x)$ defined implicitly by equation $x^2 + y^2 = 1$. Use implicit differentiation to show that

$$\frac{d^2y}{dx^2} = -\frac{1}{y^3}.$$

Calculus IB: Lecture 12

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Outline

Outline

Extreme Values of a Function

When studying a function, we sometimes need to determine its largest function value or smallest function value.

Suppose we have a function f, and c is a real number in its domain D.

- $f(c)$ is called the *global maximum* (or *absolute maximum*) of f on D if $f(c) \geq f(x)$ for *all* real number x in D.
- $f(c)$ is called the *global minimum* (or *absolute minimum*) of f on D if $f(c) \leq f(x)$ for *all* real numbers x in D.
- $f(c)$ is called a *local maximum* (or *relative maximum*) of f on D if $f(c) \geq f(x)$ for *numbers x in D which are "near" c*.
- $f(c)$ is called a *local minimum* (or *relative minimum*) of f on D if $f(c) \leq f(x)$ for *numbers x in D which are "near" c*.
- An *extremum* (or extreme value) is either a maximum or minimum, absolute or local.

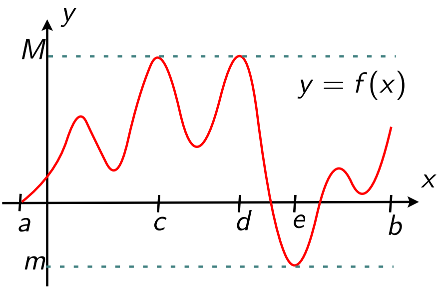
Precise definition of local maximum: Suppose we have a function f, and c is a number in its domain D. $f(c)$ is called a *local maximum* (or *relative maximum*) of f on D if there exists an $\delta > 0$ such we have $f(c) \geq f(x)$ whenever $|x - c| < \delta$ holds for any x in domain D.

Precise definition of local minimum: Suppose we have a function f, and c is a number in its domain D. $f(c)$ is called a *local minimum* (or *relative minimum*) of f on D if there exists an $\delta > 0$ such we have $f(c) \leq f(x)$ whenever $|x - c| < \delta$ holds for any x in domain D.

Note also that any global maximum (global minimum respectively) is automatically a local maximum (local minimum respectively).

Extreme Values of Functions

Here is a simple graph of $y = f(x)$ defined on a closed interval $[a, b]$ to illustrate the usage of these terms above:



The global maximum is reached at two points, $M = f(c) = f(d)$, and the global minimum is reached at one point, $m = f(e)$.

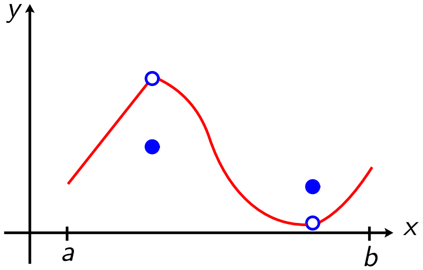
In addition, there are three other local maximum, and four other local minimum.

Extreme Values of Functions

A function may not have maximum or minimum though. Just look at the graph below:

You may run along the graph to get closer and closer to the two "holes" (y-values), but can never reach those heights as any function value.

This is obviously due to the existence of points of discontinuity.



We are more interested in extreme values of continuous functions.

The Extreme Value Theorem

Theorem (Extreme Value Theorem)

If f is continuous on a closed interval, then f attains an global maximum $f(c)$ and an global minimum $f(d)$ at some numbers c and d in $[a, b]$.

The global maximum/minimum may be reached at the boundary points of the closed interval $[a, b]$, or at points inside the open interval (a, b).

If $f(c)$ is a local maximum for some c in (a, b), then $f(c+h) \leq f(c)$ and $f(c-h) \leq f(c)$ for all sufficiently small $h > 0$. If $f'(c)$ exists, we have

$$0 \leq \lim_{h \to 0^+} \frac{f(c-h) - f(c)}{(c-h) - c} = f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{(c+h) - c} \leq 0,$$

i.e., $f'(c) = 0$.

A number c in the domain of f is called a *critical number* or *critical point* if either $f'(c) = 0$ or $f'(c)$ does not exist.

Finding Global Maximum/Minimum

Theorem (Fermat's Theorem)

If f has a local maximum or local minimum at an interior point c, and if $f'(c)$ exists, then $f'(c) = 0$.

As a result, we obtain a basic approach to find the global maximum and minimum of a differentiable function f on a closed interval $[a, b]$ is:

1. Find all critical points of f in (a, b), and the respective function values.
2. Find the function values of f at the boundary points of the interval $[a, b]$.
3. Just compare these function values above to find the largest (global maximum) and smallest (global minimum).

Example

Find global maximum/minimum of function $f(x) = x^3$ on interval $[-2, 3]$.

We can verify 0 is unique critical point of f since

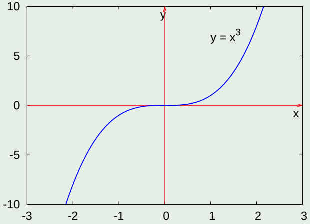
$$f'(x) = 3x^2 = 0 \iff x = 0.$$

A straightforward comparison of function values leads to

$$f(-2) = -8,$$

$$f(0) = 0,$$

$$f(3) = 27.$$



Hence $f(-2) = -8$ global minimum and $f(3) = 27$ is global maximum, but $f(0) = 0$ is neither local maximum value nor local minimum value.

Finding Global Maximum/Minimum

Example

Find the absolute maximum and minimum values of $f(x) = 3x(4 - \ln x)$ on the interval $\left[1, e^5\right]$.

We desire to find the critical point of f by its derivative:

$$f'(x) = 3(4 - \ln x) + 3x(-\frac{1}{x}) = 9 - 3\ln x \qquad \text{(Product Rule)}$$
$$f'(x) = 0 \iff \ln x = 3 \iff x = e^3$$

At the only critical point $x = e^3$: $f(e^3) = 3e^3(4 - \ln e^3) = 3e^3$.

At the endpoints of the interval $[1, e^5]$:

$$f(1) = 3(4 - \ln 1) = 12, \qquad f(e^5) = 3e^5(4 - \ln e^5) = -3e^5.$$

Then, the global maximum is $3e^3$, and the global minimum is $-3e^5$.

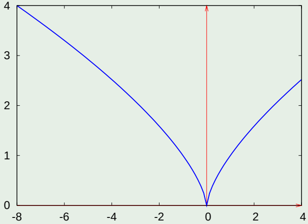
Example

Find the absolute maximum and minimum values of $f(x) = x^{\frac{2}{3}}$ on the interval $[-8, 4]$.

The derivative of f is

$$f'(x) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3x^{-\frac{1}{3}}},$$

which can never be 0.



However, $x = 0$ is a critical point since $f'(0)$ does not exist. Hence, this point also should be taken into consideration.

Compare function values at the critical point and endpoints:

$$f(0) = 0, \qquad f(-8) = (-8)^{\frac{2}{3}} = 4, \qquad f(4) = 4^{\frac{2}{3}} = \sqrt[3]{16}.$$

The absolute maximum value is 4, and the absolute minimum value is 0.

Outline

Rolle's Theorem

Combining the extreme value theorem and Fermat's theorem, it is easy to conclude Rolle's theorem.

Theorem (Extreme Value Theorem)

If f is continuous on a closed interval, then f attains an global maximum $f(c)$ and an global minimum $f(d)$ at some numbers c and d in $[a, b]$.

Theorem (Fermat's Theorem)

If f has a local maximum or local minimum at an interior point c, and if $f'(c)$ exists, then $f'(c) = 0$.

Theorem (Rolle's Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and $f(a) = f(b)$ and $a < b$, then $f'(c) = 0$ for some number $c \in (a, b)$.

Rolle's Theorem

Theorem (Rolle's Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), $f(a) = f(b)$ and $a < b$, then $f'(c) = 0$ for some number $c \in (a, b)$.

Proof.

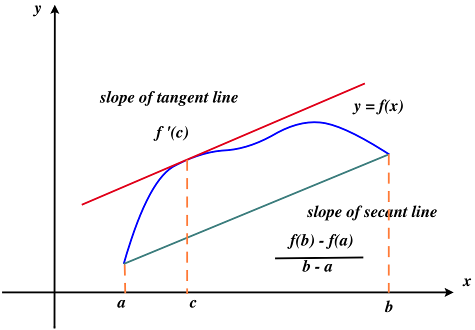
The idea is pretty simple: either $f(x) = f(a) = f(b)$ for all x in $[a, b]$, and thus $f'(x) = 0$ for all x in (a, b), or a maximum is reached at some c in (a, b) by the extreme value theorem so that $f'(c) = 0$ by the Fermat's theorem. □

Theorem (Mean Value Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

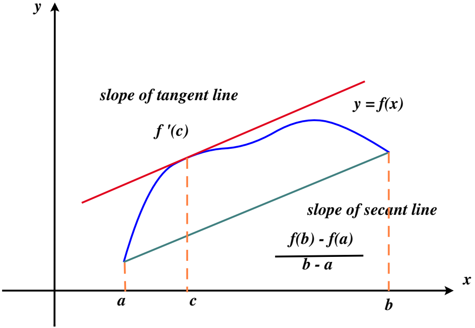
for some $c \in (a, b)$, or equivalently $f(b) - f(a) = f'(c)(b - a)$.



Mean Value Theorem

The proof of mean value theorem is based on the gap between the graph of f and the secant line joining the endpoints of the graph

$$h(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right]$$



Mean Value Theorem

The proof of mean value theorem is based on the gap between the graph of f and the secant line joining the endpoints of the graph

$$h(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right]$$

It is easy to verify $h(a) = h(b) = 0$, then $h(x)$ satisfies the conditions of Rolle's theorem. Hence, there exists some c in (a, b) such that $h'(c) = 0$:

$$h'(c) = f'(c) - \left[0 + \frac{f(b) - f(a)}{b - a}\right] = 0$$

$$\Longrightarrow f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{or} \quad f(b) - f(a) = f'(c)(b - a)$$

Mean Value Theorem

Here are some consequences of the mean value theorem:

- If $f' = 0$ on the whole interval (a, b), then f is a constant function on the interval. (for any $a < x_1 < x_2 < b$, we have $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) = 0$, i.e., $f(x_1) = f(x_2)$)
- If $f'(x) > 0$ for all x in an interval (a, b), then $f(x)$ is an increasing function on (a, b). (for any $a < x_1 < x_2 < b$, we have $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$, for some c between x_1 and x_2, i.e., $f(x_2) > f(x_1)$ since $f'(c) > 0$)
- If $f'(x) < 0$ for all x in an interval (a, b), then $f(x)$ is an decreasing function on (a, b).

Another Mean Value Theorem

Exercise

If f is twice differentiable on (a, b) and continuous on $[a, b]$, then

$$f(b) - f(a) - f'(a)(b-a) = \frac{f''(c)}{2}(b-a)^2$$

for some $c \in (a, b)$.

Hint: Consider

$$h(x) = f(x) - f(a) - f'(a)(x-a) - \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}(x-a)^2$$

$$h'(x) = f'(x) - f'(a) - 2 \cdot \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}(x-a)$$

What if you now apply the Rolle's Theorem? What is $h''(x)$?

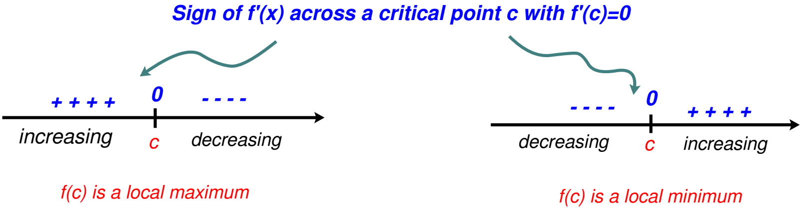
Outline

Using 1st and 2nd Derivatives in Graphing

A lot about function $y = f(x)$ can be found by $f'(x)$ and $f''(x)$.

$$
f(x) \begin{cases} f'(x) \begin{cases} \text{Critical Points : } f'(x) = 0 \text{ (or undefined)} \\ \text{Intervals of Increase/Decrease } (\pm \text{ sign of } f'(x)) \\ \text{1st Derivative Test for Local Extrema} \\ \text{(i.e., look at the sign line of } f'.) \end{cases} \\ f''(x) \begin{cases} \text{Convex/Concave Intervals } (\pm\text{sign of } f''(x)) \\ \text{Inflection Points - where concavity changes.} \\ \text{2nd Derivative Test for Local Extrema} \end{cases} \end{cases}
$$

First Derivative Test



$f(c)$ is neither a local maximum nor a local minimum if the sign of f' does not change across c.

Second Derivative Test

Second order derivative test

$$f'(c) = 0 \text{ and } \begin{cases} f''(c) < 0, & \text{then } f(c) \text{ is a local maximum.} \\ f''(c) > 0, & \text{then } f(c) \text{ is a local minimum.} \end{cases}$$

This follows from the limit definition of derivative and the 1st derivative test. If $f'(c) = 0$ and $f''(c) > 0$, then we have

$$\lim_{h\to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h\to 0} \frac{f'(c+h) - 0}{h} = f''(c) > 0$$

which means when h is very close to 0, $\dfrac{f'(c+h)}{h} \approx f''(c) > \dfrac{1}{2}f''(c) > 0$.

Hence if $h > 0$, we have $f'(c+h) > \frac{h}{2}f''(c) > 0$; and if $h < 0$, we have $f'(c+h) < \frac{h}{2}f''(c) < 0$. Thus $f(c)$ is a local minimum by the 1st derivative test.

Second Derivative Test

How to obtain $\frac{f'(c+h)}{h} \approx f''(c) > \frac{1}{2}f''(c) > 0$ from the limit?

The formal description: there exists $\delta > 0$ such that for any $0 < |h| < \delta$, we have

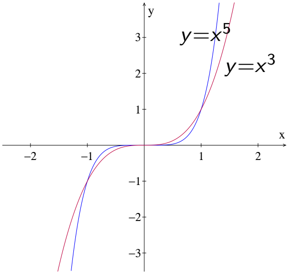
$$\frac{f'(c+h)}{h} > 0.$$

Try to prove it by (ε, δ)-definition of limit and local minimum/maximum.

Second Derivative Test

Please note that

- $f'(c) = 0$ and $f''(c) \geq 0$ does not mean $f(c)$ is a local minimum
- $f'(c) = 0$ and $f''(c) \leq 0$ does not mean $f(c)$ is a local maximum



$f(0)$ is neither local minimum nor local maximum

Second Derivative Test

Suppose f, f' and f'' are well defined on (a, b) and c in (a, b). Note that sufficient condition and necessary condition of local extrema are different.

- $f'(c) = 0$ and $f''(c) > 0$ mean $f(c)$ is a local minimum
- $f(c)$ is a local minimum means $f'(c) = 0$ and $f''(c) \geq 0$
- $f'(c) = 0$ and $f''(c) < 0$ mean $f(c)$ is a local maximum
- $f(c)$ is a local maximum means $f'(c) = 0$ and $f''(c) \leq 0$

Second Derivative Test

Exercise

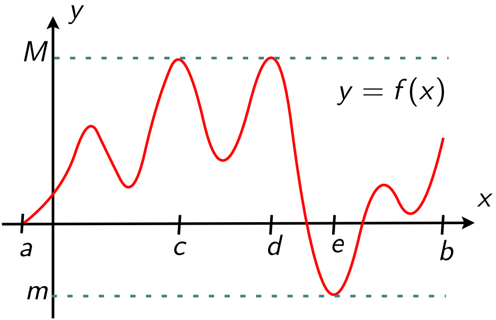
Under conditions in last page, prove $f(c)$ is a local minimum means $f'(c) = 0$ and $f''(c) \geq 0$.

Hint: If $f(c)$ is a local minimum for c in (a, b), then first derivative test means $f'(c) = 0$. Hence, we only need to show $f''(c) \geq 0$. We suppose $f''(c) < 0$, then the exercise on page 18 will lead to contradiction.

The result $f''(c) \geq 0$ cannot be improved to $f''(c) > 0$. We just need consider constant function $f(x) = c$. Every $f(x)$ are local minimum or maximum and $f'(x) = 0$, but $f''(0) = 0$ is not strictly greater than 0.

Endpoints and Local Minimum/Maximum

If $f(x)$ is differentiable on (a, b) and continuous on $[a, b]$, can we conclude the endpoints $f(a)$ (or $f(b)$) must be a local minimum or local maximum?



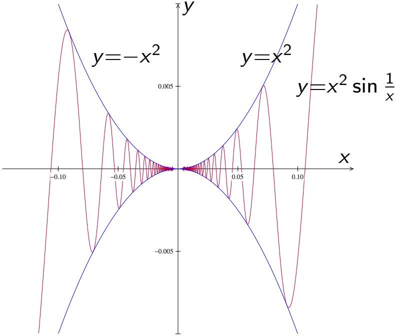
It is incorrect!

Endpoints and Local Minimum/Maximum

Consider the example we have mentioned in section chain rule:

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Here we restrict the domain as $[0, 1]$. What about $x = 0$?



Endpoints and Local Minimum/Maximum

Note that $g(0)$ is neither local minimum or local maximum of

$$g(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } 0 < x \leq 1, \\ 0 & \text{if } x = 0. \end{cases}$$

Examples: $f(x) = x^3 - 2x^2 + x - 1$

Find the intervals of increase/decrease, and local extrema of the function $f(x) = x^3 - 2x^2 + x - 1$.

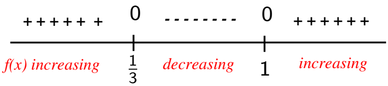
By differentiating the function, we have

$$f'(x) = 3x^2 - 4x + 1 = (3x-1)(x-1).$$

The critical points can be found by solving the equation $f'(x) = 0$:

$$(3x-1)(x-1) = 0 \Longleftrightarrow x = \frac{1}{3}, \text{ or } x = 1.$$

Let's check the sign of $f'(x)$ in intervals $x < \frac{1}{3}$, $\frac{1}{3} < x < 1$, and $x > 1$:



By the first derivative test, we have

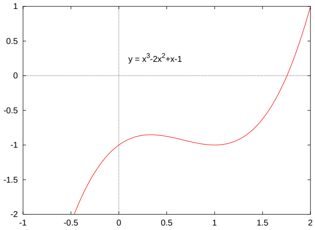
$$f\left(\frac{1}{3}\right) = -\frac{23}{27} \text{ is a local maximum and } f(1) = -1 \text{ is a local minimum.}$$

Examples: $f(x) = x^3 - 2x^2 + x - 1$

If you prefer using a table instead of a sign line:

interval	$x < \frac{1}{3}$	$x = 0$	$0 < x < 1$	$x = 1$	$1 < x$
$f'(x)$	$+ve$	0	$-ve$	0	$+ve$
$f(x)$	increasing	local max	decreasing	local min	increasing

The graph of $y = x^3 - 2x^2 + x - 1$:



No global extrema:

$$\lim_{x \to +\infty} f(x) = +\infty$$

$$\lim_{x \to -\infty} f(x) = -\infty$$

Examples: $f(x) = (x^2 - 3)e^{-x}$

Find the intervals of increase/decrease, and local extrema of the function $f(x) = (x^2 - 3)e^{-x}$.

We have

$$\begin{aligned} f'(x) = & e^{-x}\frac{d(x^2-3)}{dx} + (x^2-3)\frac{de^{-x}}{dx} \\ = & 2xe^{-x} - (x^2-3)e^{-x} = -(x^2-2x-3)e^{-x}, \end{aligned}$$

then the critical points are:

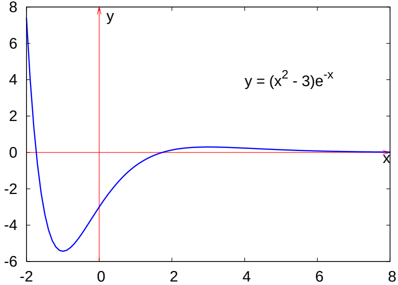
$$-(x^2 - 2x - 3)e^{-x} = 0 \iff (x-3)(x+1) = 0 \iff x = -1 \text{ or } x = 3$$

interval	$x < -1$	$x = -1$	$-1 < x < 3$	$x = 3$	$3 < x$
$f'(x)$	$-ve$	0	$+ve$	0	$-ve$
$f(x)$	decreasing	local min	increasing	local max	decreasing

local minimum: $f(-1) = -2e$ local maximum: $f(3) = 6e^{-3}$

Examples: $f(x) = (x^2 - 3)e^{-x}$

Graph of $f(x) = (x^2 - 3)e^{-x}$:



Note that the graph suggests $\lim_{x \to +\infty} (x^2 - 3)e^{-x} = 0$. We will discuss this type of limits in more details when discussing l'Hôpital's Rule later.

The local minimum $f(-1) = -2e$ is actually an global minimum.

Calculus IB: Lecture 13

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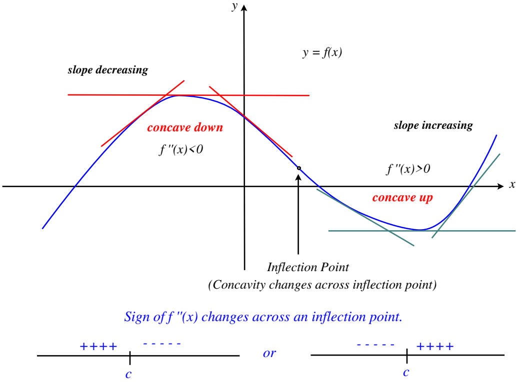
Outline

Outline

Convexity/Concavity and 2nd Derivatives

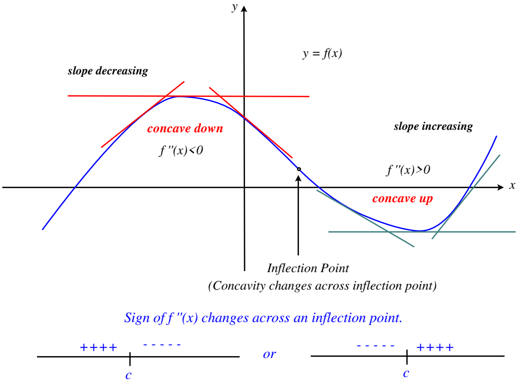
What does the graph of $y = f(x)$ on an interval mean by the sign of f''?

- $f'' > 0 \Longrightarrow f'$ is increasing (the slope of tangent line is increasing) $\Longrightarrow f$ is concave up (or strictly convex)
- $f'' < 0 \Longrightarrow f'$ is decreasing (the slope of tangent line is decreasing). $\Longrightarrow f$ is concave down (or strictly concave)



Convexity/Concavity and 2nd Derivatives

If concavity (up/down) on both sides of a point $(c, f(c))$ on the graph of the function $y = f(x)$, where f is continuous, are different, then the point is called a point of inflection.



Convexity/Concavity and 2nd Derivatives

The convexity of f on an interval:

- $f'' \geq 0 \implies f$ is convex
- $f'' > 0 \implies f$ is strictly convex (concave up)
- $f'' \geq c$ for some $c > 0 \implies f$ is strongly convex

The concavity of f on an interval:

- $f'' \leq 0 \implies f$ is concave
- $f'' < 0 \implies f$ is strictly concave (concave down)
- $f'' \leq c$ for some $c < 0 \implies f$ is strongly concave

The linear function is both convex and concave. Since we have $f''(x) = 0$ for $f(x) = ax + b$.

A strictly convex function may NOT be a strongly convex function.

Consider the function $f(x) = e^x$ defined on $(-\infty, \infty)$, we have

$$f'(x) = e^x, \quad f''(x) = e^x \quad \text{and} \quad \lim_{x \to -\infty} f''(x) = 0.$$

Since $f''(x) > 0$, the function $f(x)$ is strictly convex on $(-\infty, \infty)$.

However, for any $c > 0$, there exists M such that $f''(x) < c$ for any $x < M$. Hence, $f(x)$ is not strongly convex.

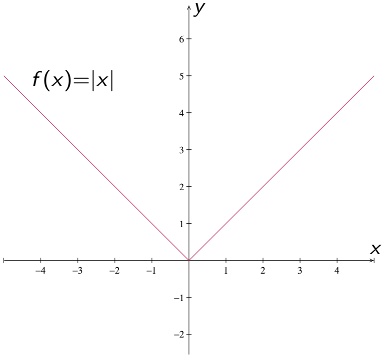
Now we consider the function $f(x) = e^x$ defined on $[a, \infty)$, then

$$f''(x) = e^x \geq e^a$$

By taking $c = e^a$, we have $f''(x) \geq c$ for any x in $[a, \infty)$, which means the function is strongly convex.

Convex/Concave Functions

We can also define of convexity/concavity for non-differentiable functions.



A real-valued function defined on an interval is called convex (concave) if the line segment between any two points on the graph of the function lies above (below) the graph between the two points.

Outline

Graph Sketching

Roughly speaking, 1st/2nd derivatives of a function f, together with some limits (asymptotes), symmetric properties, and intercepts, can help determine the shape of the graph of f pretty well.

We can also use software (e.g. MATLAB) to plot the figure of a function (not allowed in final exam).

Graph Sketching

The following strategies can be used in final exam:

1. Identify the domain of f, and any symmetry property of the graph: e.g., even, odd function?,. . . .
2. Identify the asymptotes, either vertical or horizontal.
3. Compute the first and second derivative : f', f''.
4. Determine the critical points (where is $f'(x) = 0$, or $f'(x)$ does not exist), and interval of increase/decrease (i.e., where does f have positive rate of change $f'(x) > 0$, or negative rate of change $f'(x) < 0$), by looking at the sign line of f'.
5. Determine the concavity of the graph by the sign line of f'', and indicate the inflection points.
6. Plot a suitable number of points, especially the x and/or y intercept, local max/min points, inflection points.

Graph Sketching: Example

Sketch the graph of the following function

$$y = f(x) = \frac{x^2 - 3}{x^3}.$$

The domain is $x \neq 0$. Note that $x = 0$ is a vertical asymptote since

$$\lim_{x \to 0^+} \frac{x^2 - 3}{x^3} = -\infty, \quad \lim_{x \to 0^-} \frac{x^2 - 3}{x^3} = \infty.$$

and $y = 0$ is a horizontal asymptote since

$$\lim_{x \to \infty} \frac{x^2 - 3}{x^3} = 0 = \lim_{x \to -\infty} \frac{x^2 - 3}{x^3}$$

The function is an odd since $f(-x) = -f(x)$.

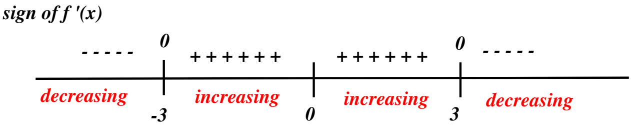
The x-intercept is $x = \pm\sqrt{3}$, since $f(\pm\sqrt{3}) = 0$.

Graph Sketching: Example

Compute the 1-st derivatives and their sign lines:

$$
\begin{aligned}
f'(x) &= \frac{x^3 \frac{d}{dx}(x^2-3) - (x^2-3)\frac{d}{dx}x^3}{x^6} \\
&= \frac{x^2(9-x^2)}{x^6} \\
&= \frac{(9-x^2)}{x^4} = \frac{(3-x)(3+x)}{x^4} \\
\Longrightarrow &\quad \text{critical point: } x = \pm 3.
\end{aligned}
$$

The interval of increasing/decreasing of f from the sign line of f':



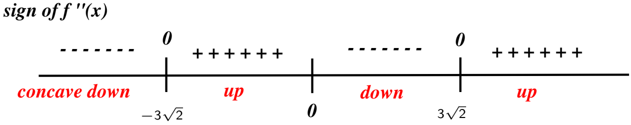
Graph Sketching: Example

Compute the 2-st derivatives and their sign lines:

$$
\begin{aligned}
f''(x) &= \frac{x^4(-2x) - (9 - x^2)(4x^3)}{x^8} \\
&= \frac{2(x^2 - 18)}{x^5} \\
&= \frac{2(x - 3\sqrt{2})(x + 3\sqrt{2})}{x^5}
\end{aligned}
$$

$\Longrightarrow$ Inflection points: $\left(3\sqrt{2}, f(3\sqrt{2})\right)$ and $\left(-3\sqrt{2}, f(-3\sqrt{2})\right)$.

The concavity of f can be found easily from the sign line of f'':



Graph Sketching: Example

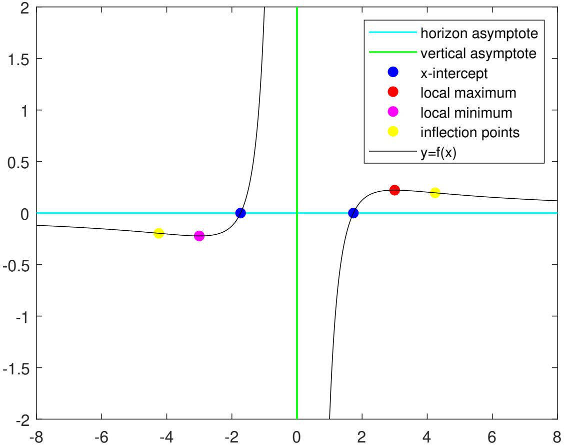
Putting together all the information above

- vertical asymptote: y-axis
- horizontal asymptote: x-axis
- x-intercept: $x = \pm\sqrt{3}$
- local minimum: $f(-3) = -\dfrac{2}{9}$
- local maximum: $f(3) = \dfrac{2}{9}$
- inflection points: $\left(-3\sqrt{2}, -\dfrac{5\sqrt{2}}{36}\right), \left(3\sqrt{2}, \dfrac{5\sqrt{2}}{36}\right)$

and plotting a suitable number of points, we can sketch the graph.

Graph Sketching: Example

$$y = f(x) = \frac{x^2 - 2}{x^3}$$



Graph Sketching by MATLAB

```
% crate the figure
figure;

% enumerate point: -8, -7.99 .... 7.99, 8
x = -8: 0.01: 8;

% compute f(x): f(-8), f(-7.99) .... f(7.99), f(8)
y = (x.^2 - 3)./x.^3;

% plot graph by connecting
% (-8, f(-8)), (-7.99, f(-7.99)) ... (8, f(8)), (-8, f(-8))
plot(x, y, 'k-');  hold on;

% set the range of display
xlim([-8, 8]); ylim([-2, 2]);

% draw x-axis and y-axis
plot(xlim, [0,0], 'b', 'LineWidth', 1); hold on;
plot([0,0], ylim, 'r', 'LineWidth', 1); hold on;

% add the legends
legend('y=f(x)', 'x-axis', 'y-axis')
```

Calculus IB: Lecture 14

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Outline

Outline

1. Optimization Problems
2. Characterization of Convexity/Concavity
3. Properties of Convex Functions

Optimization Problems

Mathematical optimization (alternatively spelled optimisation) is the selection of a best element (with regard to some criterion) from some set of available alternatives.

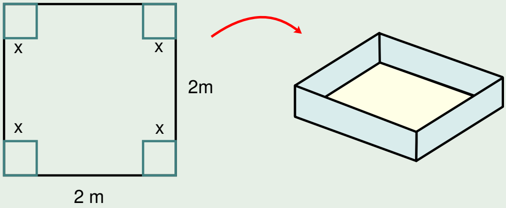
Optimization problems of sorts arise in all quantitative disciplines from computer science and engineering to operations research and economics, and the development of solution methods has been of interest in mathematics for centuries.

In the simplest case, an optimization problem consists of finding the maximum or minimum of a certain function to meet certain purpose, depending on the requirement of the problem.

Optimization Problems

Example (volume of the box)

If an open-top box is to be made by cutting four square corners from a 2m by 2m sheet of tin, what is the largest possible volume of the box?



We write the volume of the box as a function of x:

$$V(x) = x(2-2x)^2 = 4x - 8x^2 + 4x^3$$

where $0 \leq x \leq 1$. We want to find the maximum of V.

Optimization Problems

Example (volume of the box)

We first find the critical point of $V(x) = 4x - 8x^2 + 4x^3$:

$$\frac{dV}{dx} = 4 - 16x + 12x^2 = 0 \Longleftrightarrow x = 1 \quad \text{or} \quad x = \frac{1}{3}.$$

By comparing the function values at critical points and endpoints

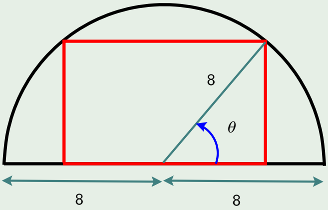
$$V(1) = 0, \quad V(0) = 0, \quad V\left(\frac{1}{3}\right) = \frac{16}{27},$$

the maximum volume is $\frac{16}{27}\text{m}^3$, which reached by cutting $\frac{1}{3}\text{m}$.

Optimization Problems

Example (rectangle in semi-circle)

What is the largest area of rectangle inscribed in a semi-circle of radius 8?



The area of the inscribed rectangle as a function of the angle θ, where $0 \le \theta \le \dfrac{\pi}{2}$ is:

$$A(\theta) = 2(8\cos\theta)(8\sin\theta) = 128\cos\theta\sin\theta = 64\sin 2\theta,$$

where we use the identity $\sin 2\theta = 2\sin\theta\cos\theta$.

Optimization Problems

Example (rectangle in semi-circle)

We first find the critical point of $A(\theta) = 64 \sin 2\theta$:

$$A'(\theta) = 128 \cos 2\theta = 0 \iff 2\theta = \frac{\pi}{2} \iff \theta = \frac{\pi}{4}.$$

By comparing the function values at critical points and endpoints

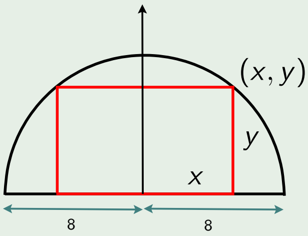
$$A(0) = 0 = A\left(\frac{\pi}{2}\right), \qquad A\left(\frac{\pi}{4}\right) = 64,$$

the maximum area is 64, which reached when $\theta = \dfrac{\pi}{4}$ and the rectangle is a square of base length $2 \cdot \left(8 \cos \frac{\pi}{4}\right) = 8\sqrt{2}$.

Example (rectangle in semi-circle)

We can also formulate the problem with

$$A(x) = 2xy = 2x\sqrt{8^2 - x^2}, \text{ where } 0 \le x \le 8.$$



We find the critical point by computing the derivative

$$\frac{dA}{dx} = 2\sqrt{64 - x^2} - \frac{2x^2}{\sqrt{64 - x^2}} = \frac{128 - 4x^2}{\sqrt{64 - x^2}} = 0 \Longrightarrow x = \sqrt{32} = 4\sqrt{2}.$$

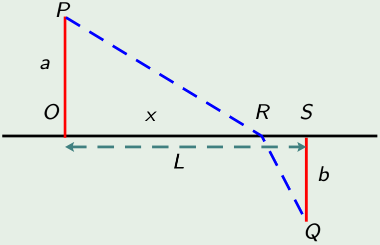
The maximum area is reached at $x = \sqrt{32} = 4\sqrt{2}$, i.e., $A(4\sqrt{2}) = 64$.

Example (Snell's Law)

The ground surfaces on two sides of a certain boundary line are made of different materials.

A man can run with speed v_1 m/s on the side of location P which a m from the line, and v_2 m/s on the side of location Q which is b m from the line.

Running along a straight path on either side, how should he choose the cross-boundary point R in order to reach location Q as soon as possible?



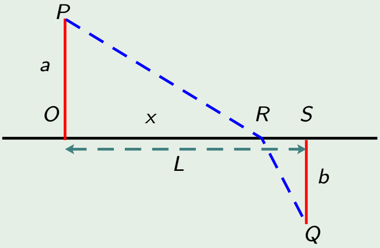
Optimization Problems

Example (Snell's Law)

Let $OR = x$, $OS = L$, then $RS = L - x$. The time to reach Q is the function as follows:

$$T(x) = \frac{\sqrt{x^2 + a^2}}{v_1} + \frac{\sqrt{(L-x)^2 + b^2}}{v_2}, \qquad \text{where } 0 \le x \le L.$$

We want to select x in $[0, L]$ to minimize $T(x)$.



Optimization Problems

Example (Snell's Law)

We first consider the critical points of $T(x)$:

$$T = \frac{\sqrt{x^2 + a^2}}{v_1} + \frac{\sqrt{(L-x)^2 + b^2}}{v_2} \qquad \text{where } 0 \le x \le L$$

$$\frac{dT}{dx} = \frac{x}{v_1\sqrt{x^2 + a^2}} + \frac{-(L-x)}{v_2\sqrt{(L-x)^2 + b^2}}$$

$$\frac{dT}{dx} = 0 \iff \frac{x}{v_1\sqrt{x^2 + a^2}} = \frac{L-x}{v_2\sqrt{(L-x)^2 + b^2}}$$

Solving the equation by squaring it will lead to a 4-th degree equation which is so complicated.

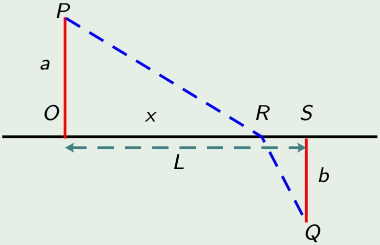
We can also look at its geometric meaning and construct a simple relationship.

Example (Snell's Law)

Let $\angle OPR = \alpha$, $\angle RQS = \beta$. Then we can simplify the condition of $T'(x) = 0$ as follows:

$$\frac{x}{v_1\sqrt{x^2+a^2}} = \frac{L-x}{v_2\sqrt{(L-x)^2+b^2}} \Longleftrightarrow \frac{1}{v_1}\cdot\frac{OR}{PR} = \frac{1}{v_2}\cdot\frac{RS}{QR}$$

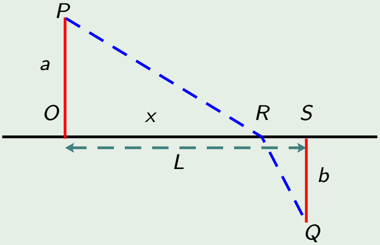
$$\Longleftrightarrow \frac{1}{v_1}\cdot\sin\alpha = \frac{1}{v_2}\cdot\sin\beta \Longleftrightarrow \frac{\sin\alpha}{\sin\beta} = \frac{v_1}{v_2}$$



Example (Snell's Law)

Let $\angle OPR = \alpha$, $\angle RQS = \beta$. Hence, the condition of $T'(x) = 0$ can be written as

$$\frac{\sin\alpha}{\sin\beta} = \frac{v_1}{v_2}.$$



Can we conclude the x leads to $T'(x) = 0$ is a minimizer?

Example (Snell's Law)

We should check the sign of $T''(x)$:

$$T''(x) = \frac{a^2}{v_1\left(x^2 + a^2\right)^{\frac{3}{2}}} + \frac{b^2}{v_2\left((L-x)^2 + a^2\right)^{\frac{3}{2}}} > 0.$$

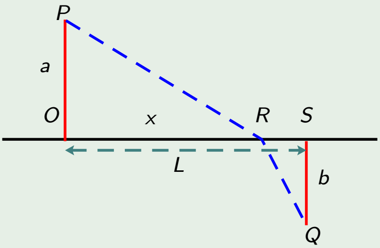
Hence, the function $T'(x)$ is increasing on $(-\infty, +\infty)$.

Let x_0 be the critical point of T such that $T'(x_0) = 0$. Then, we have $T'(x) < T(x_0) = 0$ for all $x < x_0$ and $T'(x) > T(x_0) = 0$ for all $x > x_0$, which implies $f(x_0)$ is the unique global minimum.

Example (Snell's Law)

Let $\angle OPR = \alpha$, $\angle RQS = \beta$. The point R should be the one satisfies

$$\frac{\sin \alpha}{\sin \beta} = \frac{v_1}{v_2}.$$

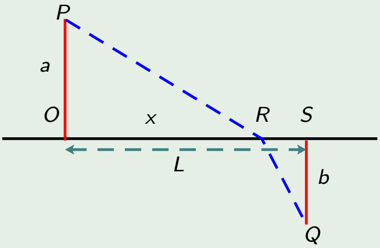


Optimization Problems

Example (Snell's Law)

Let $\angle OPR = \alpha$, $\angle RQS = \beta$. In optics, Snell's law says if a light passing through boundary OS between two different isotropic media (such as water, glass, or air) from P with directions PR, its trace satisfies

$$\frac{\sin\alpha}{\sin\beta} = \frac{v_1}{v_2}.$$

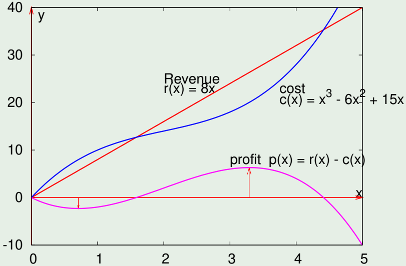


Optimization Problems

Example (maximizing the profit)

Suppose $c(x) = x^3 - 6x^2 + 15x$ is the cost of producing x thousands of units of a product, and $r(x) = 8x$ the revenue of selling x thousands of units of the product.

Then $p(x) = r(x) - c(x)$ is the profit of selling x thousands of units of the product. Find the production level which maximizes the profit.



Example (maximizing the profit)

We find the critical points

$$p'(x) = r'(x) - c'(x) = 8 - (3x^2 - 12x + 15) = 0 \Longrightarrow x = 2 \pm \frac{\sqrt{15}}{3}$$

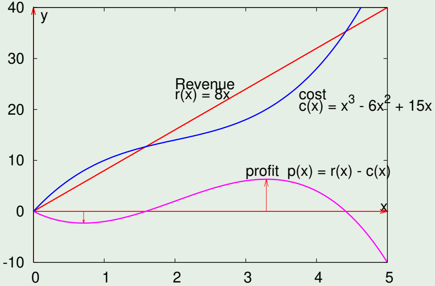
It is easy to check that $p'(x) < 0$ for $x < 2 - \frac{\sqrt{15}}{3}$, or $x > 2 + \frac{\sqrt{15}}{3}$, and $p'(x) > 0$ when $2 - \frac{\sqrt{15}}{3} < x < 2 + \frac{\sqrt{15}}{3}$.

Hence, the maximum profit is reached at $x = 2 + \frac{\sqrt{15}}{3}$.

Example (maximizing the profit)

At $x = 2 - \dfrac{\sqrt{15}}{3}$, the profit is a local minimum (local maximum loss).

Hence, you will be fired if you select $x = 2 - \dfrac{\sqrt{15}}{3}$.



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Closed Form Expression

For all above examples, we can provide the closed from expression for the solution of the optimization problems.

A closed-form expression is an expression expressed using a finite number of standard operations. It may contain

- constants, variables
- certain "well-known" operations (e.g., $+ - \times \div$)
- elementary functions (e.g., n-th root, exponent, logarithm, trigonometric functions, and inverse trigonometric functions)

It usually does not contain limit, differentiation, or integration (will be introduced in last 2 or 3 weeks).

The set of operations and functions admitted in a closed-form expression may vary with author and context.

Closed Form Expression

Unfortunately, the most of optimization problems in real-world application have no closed from expression.

Consider the simple polynomial function

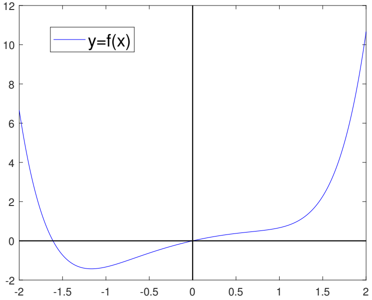
$$y = f(x) = \frac{1}{6}x^6 - \frac{1}{2}x^2 + x \ \text{ whose derivative is } \ f'(x) = x^5 - x + 1.$$

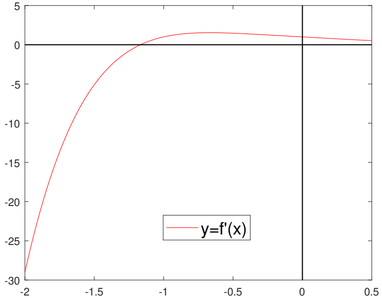
We want to find its local minimum/maximum.

Closed Form Expression

Consider that $f'(-1) = 1 > 0$ and $f'(-2) = -29 < 0$, then intermediate value theorem tell us there exists some c in $(-2, -1)$ such that $f'(c) = 0$.

We also have $f''(x) = 5x^4 - 1$. Hence $f''(c) > 0$ since $-2 < c < -1$, which means $f(c)$ is a local minimum.





Closed Form Expression

The Abel-Ruffini theorem (first asserted in 1799 and completely proved in 1824) shows that there is no closed form expression for the solution of

$$x^5 - x + 1 = 0.$$

We can use iterative methods to generate a sequence of improving approximate solutions.

For optimization problems, one class of popular functions we are interested in is convex/concave function.

Outline

Convex/Concave Functions

Let f is a real valued function defined on interval I. We call f is convex if for any x, y in I and t in $[0,1]$, it holds that

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y).$$

We call f is strictly convex if for any x,y in I $(x \neq y)$ and t in $(0,1)$, we have

$$f(tx+(1-t)y) < tf(x)+(1-t)f(y).$$

We call f is strongly convex if for any x,y in I $(x \neq y)$ and t in $(0,1)$, we have

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)-\frac{1}{2}ct(1-t)(x-y)^2$$

for some constant $c>0$ (c is independent to x, y, t and I).

The definitions of concave, strictly concave, strongly concave are similar.

1st/2nd Order Condition

Theorem (1st/2nd order condition)

Suppose function f is twice differentiable over an open interval I. Then, the following statements are equivalent:

(a) f *is convex.*

(b) $f(y) \geq f(x) + f'(x)(y - x)$, *for all* x *and* y *in* I.

(c) $f''(x) \geq 0$, *for all* x *in* I.

We say (b) is first-order condition and (b) is second order condition.

We prove this theorem by showing:

- $(a) \Longrightarrow (b)$
- $(b) \Longrightarrow (a)$
- $(b) \Longrightarrow (c)$
- $(c) \Longrightarrow (b)$

f is convex $\Longrightarrow f(y) \geq f(x) + f'(x)(y-x)$

For any $t \in [0,1]$ and x, y in I, the convexity of f means

$$f(ty + (1-t)x) \leq tf(y) + (1-t)f(x)$$

Since $ty + (1-t)x = x + t(y-x)$, we have

$$\begin{aligned} & f(x + t(y-x)) \leq tf(y) + (1-t)f(x) \\ \Longrightarrow & f(x + t(y-x)) - f(x) \leq t(f(y) - f(x)). \end{aligned}$$

If $x = y$, then

$$f(y) \geq f(x) + f'(x)(y-x) \Longleftrightarrow f(x) \geq f(x).$$

If $t = 0$, then

$$f(ty + (1-t)x) \leq tf(y) + (1-t)f(x) \Longrightarrow f(x) \leq f(x).$$

If $t = 1$, then

$$f(ty + (1-t)x) \leq tf(y) + (1-t)f(x) \Longrightarrow f(y) \leq f(y).$$

f is convex $\Longrightarrow f(y) \geq f(x) + f'(x)(y-x)$

If $x \neq y$ and $0 < t < 1$, then

$$
\begin{aligned}
& f(x + t(y-x)) - f(x) \leq t(f(y) - f(x)) \\
\Longrightarrow & f(y) - f(x) \geq \frac{f(x+t(y-x)) - f(x)}{t(y-x)} \cdot (y-x)
\end{aligned}
$$

We let $h = t(y-x)$ and take $t \to 0^+$. If $y > x$, then $h \to 0^+$ and

$$\lim_{t \to 0^+} \frac{f(x+t(y-x)) - f(x)}{t(y-x)} = \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = f'(x).$$

If $y < x$, then $h \to 0^-$ and

$$\lim_{t \to 0^+} \frac{f(x+t(y-x)) - f(x)}{t(y-x)} = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = f'(x).$$

Hence, we have

$$f(y) - f(x) \geq f'(x)(y-x).$$

Combing all above analysis, we have proved $(a) \Longrightarrow (b)$ in the theorem.

Theorem (1st/2nd order condition)

Suppose function f is twice differentiable over an open interval I. Then, the following statements are equivalent:

(a) *f is convex.*

(b) $f(y) \geq f(x) + f'(x)(y-x)$, *for all x and y in I.*

(c) $f''(x) \geq 0$, *for all x in I.*

Then we want to show that $(b) \Longrightarrow (a)$.

$f(y) \geq f(x) + f'(x)(y-x) \implies f$ is convex

The statement (b) means for any x, y and z in I, we have

$$\begin{aligned} f(x) \geq & f(z) + f'(z)(x-z) \\ f(y) \geq & f(z) + f'(z)(y-z) \end{aligned}$$

Let $z = tx + (1-t)y$, then z is in I for any $0 \leq t \leq 1$. We have

$$\begin{aligned} & tf(x) + (1-t)f(y) \\ \geq & tf(z) + tf'(z)(x-z) + (1-t)f(z) + (1-t)f'(z)(y-z). \end{aligned}$$

On the other hand

$$\begin{aligned} t(x-z) + (1-t)(y-z) = & tx - tz + (1-t)y - (1-t)z \\ = & tx + (1-t)y - z = 0. \end{aligned}$$

Hence, $tf(x) + (1-t)f(y) \geq f(z) = f(tx + (1-t)y)$, which means (a).

f is convex $\Longrightarrow f(y) \geq f(x) + f'(x)(y-x)$

We have proved $(a) \iff (b)$ in the theorem.

Theorem (1st/2nd order condition)

Suppose function f is twice differentiable over an open interval I. Then, the following statements are equivalent:

(a) *f is convex.*

(b) $f(y) \geq f(x) + f'(x)(y-x)$, *for all x and y in I.*

(c) $f''(x) \geq 0$, *for all x in I.*

Then we want to show that $(b) \Longrightarrow (c)$ and $(c) \Longrightarrow (b)$.

$f(y) \geq f(x) + f'(x)(y-x) \Longrightarrow f''(x) \geq 0$

The first order condition (b) means for any x and y in I, we have

$$f(y) \geq f(x) + f'(x)(y-x) \Longrightarrow f'(x)(y-x) \leq f(y) - f(x).$$
$$f(x) \geq f(y) + f'(y)(x-y) \Longrightarrow f(y) - f(x) \leq f'(y)(y-x).$$

Then $f'(x)(y-x) \leq f(y) - f(x) \leq f'(y)(y-x)$.

For any $x \neq y$, we divide $(y-x)^2$ on both sides and obtain

$$\frac{f'(y) - f'(x)}{y-x} \geq 0.$$

Let $y = x + h$ and take $h \to 0$, then we have

$$f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{f'(y) - f'(x)}{(y-x)^2} \geq 0.$$

$f''(x) \geq 0 \implies f(y) \geq f(x) + f'(x)(y-x)$

Exercise

If f is twice differentiable on (a, b) and continuous on $[a, b]$, then

$$f(b) - f(a) - f'(a)(b-a) = \frac{f''(c)}{2}(b-a)^2$$

for some $c \in (a, b)$.

For any x and y in I, we have

$$f(y) - f(x) - f'(x)(y-x) = \frac{f''(z)}{2}(y-x)^2$$

for some z in $[x, y]$ (which means z is in I and $f''(z) \geq 0$). Then

$$f(y) = f(x) + f'(x)(y-x) + \frac{f''(z)}{2}(y-x)^2 \geq f(x) + f'(x)(y-x).$$

1st/2nd Conditions

Now we have presented all of proof of the theorem.

Theorem (1st/2nd order condition)

Suppose function f is twice differentiable over an open interval I. Then, the following statements are equivalent:

(a) f *is convex.*

(b) $f(y) \geq f(x) + f'(x)(y - x)$, *for all* x *and* y *in* I.

(c) $f''(x) \geq 0$, *for all* x *in* I.

We can also give 1st/2nd conditions for strictly/strongly convex function.

Now we have presented all of proof of the theorem.

Theorem (1st/2nd order condition)

Suppose function f is twice differentiable over an open interval I. Then, the following statements are equivalent:

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Now we have presented all of proof of the theorem.

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Suppose function f is twice differentiable over an open interval I. Then, the following statements are equivalent:

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(c) $f''(x) \geq 0$, *for all x in I.*

We can also give 1st/2nd conditions for strictly/strongly convex function.

Outline

Global/Local Minimum of Convex Function

Any local minimum of convex function is also a global minimum.

Theorem (also holds for non-differentiable function)

Suppose function f is convex on interval I. If x^* is is a local minimum over I, then x^* is also a global minimum of f over a convex set I.

Proof.

Since $f(x^*)$ is a local minimum, for any y in I, we can choose a sufficient small $t < 1$, such that $ty + (1-t)x^*$ in I and $f(x^*) \leq f(x^* + t(y - x^*))$. The convexity of f implies

$$f(x^*) \leq f(x^* + t(y - x^*)) = f(ty + (1-t)x^*) \leq tf(y) + (1-t)f(x^*)$$

$$\Longrightarrow f(x^*) \leq tf(y) + (1-t)f(x^*) \Longrightarrow f(x^*) \leq f(y)$$

□

First-Order Optimal Condition

Theorem

If function f is convex and differentiable over an interval I. Then any point x^* that satisfies $f'(x^*) = 0$ holds that $f(x^*)$ is a global minimum.

Proof.

The 1-st order condition of convex and differentiable function means

$$f(y) \geq f(x^*) + f'(x^*)(y - x^*) = f(x^*)$$

for all y in I. $\square$

Consider that the convex and differentiable function $f(x) = e^x$ with domain $[1, \infty)$. The minimum is $f(1) = e$ but $f'(1) = e \neq 0$.

First-Order Optimal Condition

We desire to establish an equivalent condition for global minimum of convex and differentiable function.

A good strategy is relaxing the condition of $f'(x^*) = 0$ to

$$f'(x^*)(y - x^*) \geq 0$$

holds for all y in I.

First-Order Optimal Condition

Theorem (sufficient condition)

If function f is convex and differentiable over an interval I. Then any point x^* that satisfies

$$f'(x^*)(y - x^*) \geq 0$$

for all y in I holds that $f(x^*)$ is a global minimum.

Theorem (necessary condition)

If function f is convex and differentiable over an interval I. Then for any point x^* such that $f(x^*)$ is a global minimum, we have

$$f'(x^*)(y - x^*) \geq 0$$

for all y in I.

Proof (necessary condition).

Let x^* in I and $f(x^*)$ is a global minimum. Suppose y in I such that

$$f'(x^*)(y - x^*) < 0.$$

There must hold that $y \neq x^*$. Let $t > 0$, $h = t(y - x^*)$. Taking $t \to 0^+$, then

$$\begin{aligned}
&\lim_{t \to 0^+} \frac{f(x^* + t(y - x^*)) - f(x^*)}{t} \\
=&(y - x^*) \cdot \lim_{t \to 0^+} \frac{f(x^* + t(y - x^*)) - f(x^*)}{t(y - x^*)} \\
=&(y - x^*) \cdot \lim_{h \to 0} \frac{f(x^* + h) - f(x^*)}{h} = f'(x^*)(y - x^*) < 0.
\end{aligned}$$

For sufficient small t, we have $f(x^* + t(y - x^*) - f(x^*) < 0$ for $x^* + t(y - x^*)$ in I, which contradicts to x^* is global minimum. Hence, we must have

$$f'(x^*)(y - x^*) \geq 0$$

and the convexity means $f(y) \geq f(x^*) + f'(x^*)(y - x^*) \geq f(x^*)$. $\square$

First-Order Optimal Condition

Note that the equivalent condition

$$f'(x^*)(y - x^*) \geq 0,$$

only depends on the function value and the derivative of f.

Hence, it works for any convex and differentiable even if f' is not differentiable (f'' may not exist).

Calculus IB: Lecture 15

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Outline

Outline

1. Geometric View of Convex Function
2. Global/Local Minimum of Convex Function
3. Linear Approximation

Geometric View of Convex Function

Definition (convex function)

Let f is a real valued function defined on interval I. We call f is convex if for any x_1, x_2 in I and t in $[0,1]$, it holds that

$$f(tx_1 + (1-t)x_2) \le tf(x_1) + (1-t)f(x_2).$$

Theorem (1st/2nd order condition)

Suppose function f is twice differentiable over an open interval I. Then, the following statements are equivalent:

(a) *f is convex.*

(b) *$f(x) \ge f(x_0) + f'(x_0)(x - x_0)$, for all x and x_0 in I.*

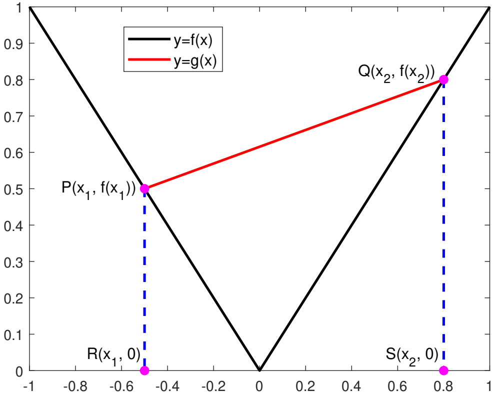
(c) *$f''(x) \ge 0$, for all x in I.*

Geometric View of Convex Function

Let $f(x)$ be a convex function and define the linear function $(x_1 \neq x_2)$

$$g(x) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + f(x_1).$$

Then we have $g(x_1) = f(x_1)$, $g(x_2) = f(x_2)$ and $g(x) \geq f(x)$ for any $x_1 \leq x \leq x_2$.



Geometric View of Convex Function

We can prove $g(x) \geq f(x)$ for any $x_1 < x < x_2$ by the convexity of f.

Proof.

Since $x_1 < x < x_2$, there exists $0 < t < 1$ such that $x = tx_1 + (1-t)x_2$. Using the definition of convexity, we have

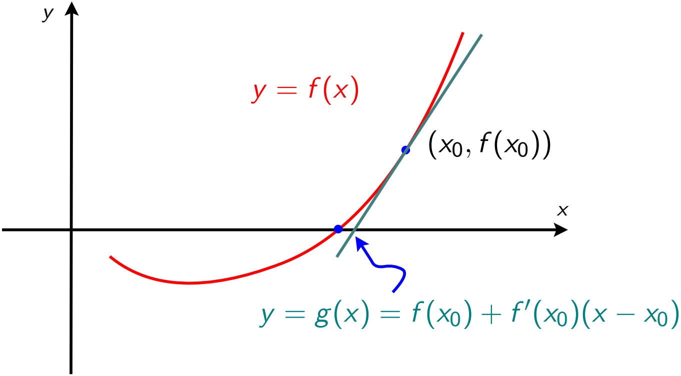
$$\begin{aligned}
g(x) =& \frac{f(x_2) - f(x_1)}{x_2 - x_1}(x - x_1) + f(x_1) \\
=& \frac{f(x_2) - f(x_1)}{x_2 - x_1}(tx_1 + (1-t)x_2 - x_1) + f(x_1) \\
=& \frac{f(x_2) - f(x_1)}{x_2 - x_1}(1-t)(x_2 - x_1) + f(x_1) \\
=& (1-t)(f(x_2) - f(x_1)) + f(x_1) \\
=& (1-t)f(x_2) + tf(x_1) \\
\geq& f(tx_1 + (1-t)x_2) = f(x)
\end{aligned}$$

$\square$

Geometric View of Convex Function

Theorem (1st order condition)

Suppose f is differentiable over an open interval I. Then f is convex is equivalent to $f(x) \geq f(x_0) + f'(x_0)(x - x_0)$ holds for all x and x_0 in I.



Outline

Global/Local Minimum of Convex Function

Any local minimum of convex function is also a global minimum.

Theorem (also holds for non-differentiable function)

Suppose function f is convex on interval I. If x^* is a local minimum over I, then x^* is also a global minimum of f over I.

Proof.

Since $f(x^*)$ is a local minimum, for any y in I, we can choose a sufficient small $t < 1$, such that $ty + (1-t)x^*$ in I and $f(x^*) \leq f(x^* + t(y - x^*))$. The convexity of f implies

$$f(x^*) \leq f(x^* + t(y - x^*)) = f(ty + (1-t)x^*) \leq tf(y) + (1-t)f(x^*)$$

$$\Longrightarrow f(x^*) \leq tf(y) + (1-t)f(x^*) \Longrightarrow f(x^*) \leq f(y)$$

□

First-Order Optimal Condition

Theorem

If function f is convex and differentiable over an interval I. Then any point x^* that satisfies $f'(x^*) = 0$ holds that $f(x^*)$ is a global minimum.

Proof.

The 1-st order condition of convex and differentiable function means

$$f(y) \geq f(x^*) + f'(x^*)(y - x^*) = f(x^*)$$

for all y in I. □

Consider that the convex and differentiable function $f(x) = e^x$ with domain $[1, \infty)$. The minimum is $f(1) = e$ but $f'(1) = e \neq 0$.

First-Order Optimal Condition

We desire to establish an equivalent condition for global minimum of convex and differentiable function.

A good strategy is relaxing the condition of $f'(x^*) = 0$ to

$$f'(x^*)(y - x^*) \geq 0$$

holds for all y in I.

First-Order Optimal Condition

Theorem (sufficient condition)

If function f is convex and differentiable over an interval I. Then any point x^* that satisfies

$$f'(x^*)(y - x^*) \geq 0$$

for all y in I holds that $f(x^*)$ is a global minimum.

Theorem (necessary condition)

If function f is convex and differentiable over an interval I. Then for any point x^* such that $f(x^*)$ is a global minimum, we have

$$f'(x^*)(y - x^*) \geq 0$$

for all y in I.

Proof (necessary condition).

Let x^* in I and $f(x^*)$ is a global minimum. Suppose y in I such that

$$f'(x^*)(y - x^*) < 0.$$

There must hold that $y \neq x^*$. Let $t > 0$, $h = t(y - x^*)$. Taking $t \to 0^+$, then

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{f(x^* + t(y - x^*)) - f(x^*)}{t} \\
=&(y - x^*) \cdot \lim_{t\to 0^+} \frac{f(x^* + t(y - x^*)) - f(x^*)}{t(y - x^*)} \\
=&(y - x^*) \cdot \lim_{h\to 0} \frac{f(x^* + h) - f(x^*)}{h} = f'(x^*)(y - x^*) < 0.
\end{aligned}$$

For sufficient small t, we have $f(x^* + t(y - x^*) - f(x^*) < 0$ for $x^* + t(y - x^*)$ in I, which contradicts to x^* is global minimum. Hence, we must have

$$f'(x^*)(y - x^*) \geq 0$$

and the convexity means $f(y) \geq f(x^*) + f'(x^*)(y - x^*) \geq f(x^*)$. □

First-Order Optimal Condition

Note that the optimal condition

$$f'(x^*)(y - x^*) \geq 0,$$

only depends on the function value and the derivative of f.

Hence, it works even if f' is not differentiable (f'' does not exist).

First-Order Optimal Condition

Exercise

Provide examples of $f(x)$ for the following cases respectively

- $f(x)$ is convex, but not differentiable
- $f(x)$ is convex and differentiable, but $f'(x)$ is non-differentiable
- $f(x)$ is convex and differentiable, and $f^{(n)}(x)$ is differentiable for all positive integer n.

Outline

Linear Approximation

Recall that the slope of the tangent line to the graph of $y = f(x)$ at $x = a$ is the derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Therefore the equation of the tangent line to the graph of $y = f(x)$ at the point $(a, f(a))$ is determined by the slope condition

$$y = f(a) + f'(a)(x - a).$$

Letting $x = a + h$, we have

$$f'(a) \approx \frac{f(a+h) - f(a)}{h} = \frac{f(x) - f(a)}{x - a} \qquad \text{when } h = x - a \approx 0$$

i.e., $f(x) \approx f(a) + f'(a)(x - a)$, when $x \approx a$. In other words,

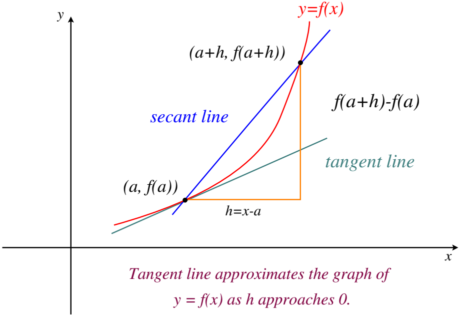
$$\text{tangent line} \xrightarrow{\textit{approximates}} \text{graph of } y = f(x) \text{ near the point } (a, f(a))$$

Linear Approximation

The *tangent line approximation at $x = a$*, or *linear approximation at $x = a$*, or *linearization at $x = a$*, of a function $y = f(x)$ (differentiable at $x = a$) is that we are using the tangent line equation (or the corresponding linear function) to approximate the given function.

$$y = f(x) \xleftarrow{\approx} \text{Tangent Line Equation : } y = f(a) + f'(a)(x - a)$$

$$\Longrightarrow f(x) \approx f(a) + f'(a)(x - a) \qquad \text{for } x - a \approx 0$$



Tangent line approximates the graph of
y = f(x) as h approaches 0.

Linear Approximation

Example ($f(x) = \sqrt{x+1}$)

Find the linear approximation of $f(x) = \sqrt{x+1}$ at $x = 0$.

We have

$$f'(x) = \frac{1}{2}(x+1)^{-1/2}, \text{ i.e. } f'(0) = \frac{1}{2}$$

and the equation of the tangent line at $(0,1)$ is

$$y = 1 + \frac{1}{2}(x-0) = 1 + \frac{1}{2}x$$

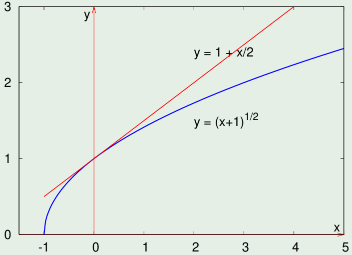
which is also called the linear approximation of $f(x) = \sqrt{x+1}$ at $x = 0$.

Linear Approximation

Example ($f(x) = \sqrt{x+1}$)

Let absolute error be $\sqrt{x+1} - 1 + \frac{x}{2}$.

x	$y = \sqrt{x+1}$	$y = 1 + \frac{x}{2}$	absolute error
0.200	1.095445	1.1000	$< 10^{-2}$
0.050	1.024695	1.0250	$< 10^{-3}$
0.005	1.002497	1.0025	$< 10^{-5}$



Example ($\sqrt[3]{8.5}$)

Find an approximate value of $\sqrt[3]{8.5}$ by the linear approximation of a suitable function.

Let $f(x) = \sqrt[3]{x} = x^{1/3}$, with $f'(x) = \frac{1}{3}x^{-2/3}$.

The linear approximation at $x = 8$ is:

$$f(x) \approx f(8) + f'(8)(x-8) = 2 + \frac{1}{12}(x-8) = \frac{x}{12} + \frac{4}{3}$$

Thus

$$f(8.5) \approx \frac{8.5}{12} + \frac{4}{3} = \frac{245}{120} = 2.04167.$$

Note that $\sqrt[3]{8.5} = 2.04093$ from a calculator.

Differential of the Function

The tangent line approximation at x is

$$f(x + \Delta x) \approx f(x) + f'(x)\Delta x,$$

where Δx denotes some increment in x (which could be negative).

Then we use Δy or Δf to denote the change in the function values

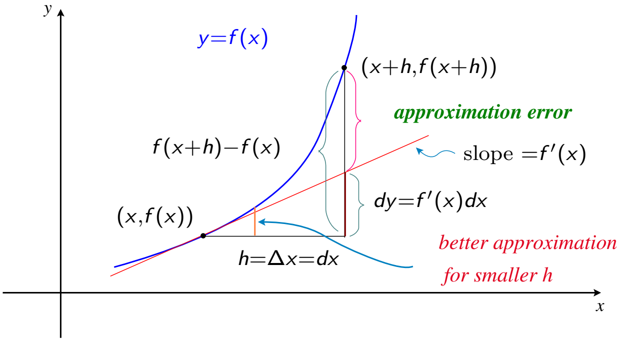
$$\Delta y = \Delta f = f(x + \Delta x) - f(x).$$

and the linear approximation be expressed as

$$\Delta f \approx f'(x)\Delta x.$$

Note that $f'(x)\Delta x$ is the change of y-value along the tangent line!

Differential of the Function



The notation of differentials $df = f'(x)dx$ is obtained by expressing Δx as dx, and $dy = df = f'(x)dx$ can be used as an approximation of

$$\Delta y = f(x + \Delta x) - f(x).$$

Linear Approximation

Example (area of a circle)

Approximate the increase in the area of a circle when the radius is increased from 10m to 10.1m.

The area of the circle is $A(r) = \pi r^2$, then $dA = A'(r)dr = 2\pi r dr$.

For $r = 10$m and $dr = 0.1$m, we have $dA = 2\pi(10)0.1 = 2\pi\text{m}^2$, which approximates the change in area ΔA.

The approximate area at $r = 10.1$ m is:

$$A \approx A(10) + dA = 100\pi + 2\pi = 102\pi\text{m}^2.$$

The exact area at $r = 10.1$m is $A = \pi(10.1)^2 = \pi(10)^2 + \Delta A$. The absolute error of the estimate is

$$|\pi(10.1)^2 - 102\pi| = 102.01\pi - 102\pi = 0.01 = |\Delta A - dy|\text{m}^2$$

Linear Approximation of Convex Function

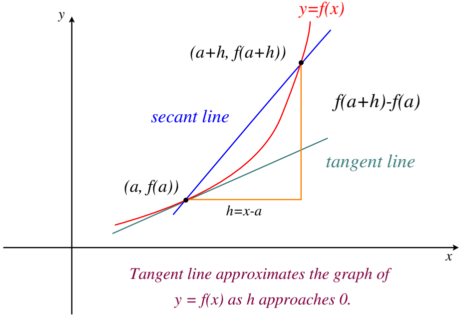
Given a differentiable function $y = f(x)$ defined on an open interval I, its linear approximation is

$$f(x) \approx f(a) + f'(a)(x - a).$$

We additionally suppose f is convex, then the first-order condition means

$$f(x) \geq f(a) + f'(a)(x - a).$$

Hence, the linear approximation provides a lower bound of convex function.



Calculus IB: Lecture 16

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Outline

Outline

L'Hôpital's Rule

One may use derivatives to help compute limits of the $\frac{0}{0}$-type or $\frac{\infty}{\infty}$-type, which is essentially what L'Hôpital's rule does.

The rule is named after the 17th-century French mathematician Guillaume de L'Hôpital, however, the theorem was first introduced in 1694 by the Swiss mathematician Johann Bernoulli.

Johann Bernoulli (1667–1748)

Guillaume de L'Hôpital (1661–1704)

L'Hôpital's Rule

Roughly speaking, if $f(a) = g(a) = 0$, we can use the idea of linear approximation to find the limit of $f(x)/g(x)$:

$$\begin{aligned}\lim_{x\to a}\frac{f(x)}{g(x)} &\approx \lim_{x\to a}\frac{f(a)+f'(a)(x-a)}{g(a)+g'(a)(x-a)}\\ &= \lim_{x\to a}\frac{f'(a)(x-a)}{g'(a)(x-a)}\\ &= \frac{f'(a)}{g'(a)}.\end{aligned}$$

In some appropriate conditions, we have

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Baby L'Hôpital's Rule

Theorem (Baby L'Hôpital's Rule, $\frac{0}{0}$-type)

Let $f(x)$ and $g(x)$ be continuous functions on an interval containing $x = a$, with $f(a) = g(a) = 0$. Suppose that f and g are differentiable, and f' and g' are continuous. Finally, suppose that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} = \frac{f'(a)}{g'(a)}.$$

We also have

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)} \quad \text{and} \quad \lim_{x \to a^-} \frac{f(x)}{g(x)} = \lim_{x \to a^-} \frac{f'(x)}{g'(x)}.$$

Baby L'Hôpital's Rule

Proof.

Since $f(a) = g(a) = 0$, we have

$$\lim_{x\to a} \frac{f(a)}{g(a)} = \lim_{x\to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x\to a} \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}$$

$$= \frac{\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}}{\lim\limits_{x\to a} \dfrac{g(x) - g(a)}{x - a}} = \frac{f'(a)}{g'(a)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

where the last step use the continuity of f' and g'. $\square$

Macho L'Hôpital's Rule

Theorem (Macho L'Hôpital's Rule, one-side)

Suppose that f and g are continuous on a closed interval $[a, b]$, and are differentiable on the open interval (a, b). Suppose that $g'(x)$ is never zero on (a, b) and $\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$ exists, and that $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0$. Then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$$

This theorem doesn't require anything about $g'(a)$, just about how g' behaves to the right of a.

The conclusion relates limit of $f(x)/g(x)$ to another one-side limit of $f(x)'/g(x)'$, and not to the value of $f'(a)/g'(a)$.

Macho L'Hôpital's Rule

Exercise/Tutorial

Prove Macho L'Hôpital's rule.

Hint: Although we do not suppose what is $f(a)$ and $g(a)$, we can define

$$F(x) = \begin{cases} 0 & x = a \\ f(x) & x > a \end{cases} \quad \textit{and} \quad G(x) = \begin{cases} 0 & x = a \\ g(x) & x > a \end{cases}.$$

Then try to prove the following theorem and apply it.

Theorem (cauchy's mean value theorem)

If $F(x)$ and $G(x)$ are continuous on $[a, b]$ and differentiable on (a, b), then there is a point c in (a, b) such that

$$(F(b) - F(a))G'(c) = (G(b) - G(a))F'(c).$$

(when $G(x) = x$, this is the same as the usual mean value theorem)

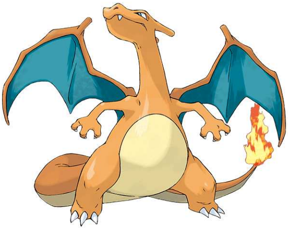
General form of L'Hôpital's Rule

The previous versions apply to forms of type $\dfrac{\infty}{\infty}$ as well as $\dfrac{0}{0}$, and apply to limits as $x \to \infty$ or $x \to -\infty$ as well as to limits $x \to a^+$ or $x \to a^-$. In all of these cases, the rule is:

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}.$$

baby form

macho form



general form

General form of L'Hôpital's Rule

Theorem (General form of L'Hôpital's Rule)

Let c and L be extended real numbers (i.e., real numbers, positive infinity, or negative infinity). Let I be an open interval containing c (for two-sided limit) or an open interval with endpoint c (for one-sided limit, or a limit at infinity if c is infinite). The real valued functions f and g are assumed to be differentiable on I except possibly at c, and additionally $g'(x) \neq 0$ on I except possibly at c. It is also assumed that

$$\lim_{x \to c} \frac{f'(x)}{g'(x)} = L.$$

If either $\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0$ or $\lim_{x \to c} |f(x)| = \lim_{x \to c} |g(x)| = \infty$, then $\lim_{x \to c} \dfrac{f(x)}{g(x)} = L$. The limits also can be one-sided limits $x \to c^+$ or $x \to c^-$, when c is a finite endpoint of I.

Examples of L'Hôpital's Rule

Example ($\frac{0}{0}$-type)

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\frac{d\sin x}{dx}}{\frac{dx}{dx}} = \lim_{x\to 0}\frac{\cos x}{1} = \cos 0 = 1.$$

However, this computation is somewhat stupid, because we have used

$$\frac{d\sin x}{dx} = \cos x.$$

We can directly obtain the result by the definition of derivative

$$\lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to 0}\frac{\sin x - \sin 0}{x - 0} = \frac{d\sin x}{dx}\bigg|_{x=0} = \cos 0 = 1.$$

Examples of L'Hôpital's Rule

Example ($\frac{0}{0}$-type)

1. $\lim_{x\to 0} \frac{x - \sin x}{x^3} = \lim_{x\to 0} \frac{1-\cos x}{3x^2} = \lim_{x\to 0} \frac{\sin x}{6x} = \frac{1}{6}$.
2. $\lim_{x\to \frac{\pi}{2}} \frac{\sec x}{1+\tan x} = \lim_{x\to \frac{\pi}{2}} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to \frac{\pi}{2}} \sin x = 1$.
3. $\lim_{x\to 0^+} \frac{\sin x}{x^2} = \lim_{x\to 0^+} \frac{\cos x}{2x} = +\infty$.
4. $\lim_{x\to 0^-} \frac{\sin x}{x^2} = \lim_{x\to 0^-} \frac{\cos x}{2x} = -\infty$.

Examples of L'Hôpital's Rule

Example ($\frac{0}{0}$-type)

1. $$\lim_{x \to 2} \frac{x^2 - 4}{x^3 - 8} = \lim_{x \to 2} \frac{\frac{d}{dx}(x^2 - 4)}{\frac{d}{dx}(x^3 - 8)} = \lim_{x \to 2} \frac{2x}{3x^2} = \frac{2 \cdot 2}{3 \cdot 2^2} = \frac{1}{3}$$

2. $$\lim_{x \to 0} \frac{x}{\sqrt{3x + 4} - 2} = \lim_{x \to 0} \frac{\dfrac{dx}{dx}}{\dfrac{d[(3x + 4)^{1/2} - 2]}{dx}}$$
$$= \lim_{x \to 0} \frac{1}{\frac{3}{2}(3x + 4)^{-1/2}} = \lim_{x \to 0} \frac{2}{3}(3x + 4)^{1/2} = \frac{4}{3}$$

Examples of L'Hôpital's Rule

Example ($\frac{\infty}{\infty}$-type)

1. $\lim_{x\to\infty}(x^2e^{-x}) = \lim_{x\to\infty}\frac{x^2}{e^x} = \lim_{x\to\infty}\frac{2x}{e^x} = \lim_{x\to\infty}\frac{2}{e^x} = 0.$

   In general $\lim_{x\to+\infty}\frac{p(x)}{e^x} = 0$ for any polynomial $p(x)$.

2. $\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+}\frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}\frac{1}{x}} = \lim_{x\to 0^+}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(-x) = 0$

3. $\lim_{x\to+\infty}\frac{\ln x}{x} = \lim_{x\to+\infty}\frac{\frac{d}{dx}\ln x}{\frac{dx}{dx}} = \lim_{x\to+\infty}\frac{\frac{1}{x}}{1} = 0$

Examples of L'Hôpital's Rule

Example ($\frac{\infty}{\infty}$-type)

1. $\lim\limits_{x\to+\infty} x^{\frac{1}{x}} = \lim\limits_{x\to+\infty} e^{\ln x^{\frac{1}{x}}} = \lim\limits_{x\to+\infty} e^{\frac{1}{x}\ln x} = e^{\lim\limits_{x\to+\infty}\frac{\ln x}{x}} = e^0 = 1$, since

$$\lim_{x\to+\infty}\frac{\ln x}{x} = 0.$$

2. $\lim\limits_{x\to+\infty}\left(1-\dfrac{4}{x}\right)^x = \lim\limits_{x\to+\infty} e^{\ln\left(1-\frac{4}{x}\right)^x} = \lim\limits_{x\to+\infty} e^{x\ln(1-\frac{4}{x})} = e^{-4}$, since

$$\lim_{x\to+\infty}\frac{\ln(1-\frac{4}{x})}{\frac{1}{x}} = \lim_{x\to+\infty}\frac{\frac{\frac{4}{x^2}}{1-\frac{4}{x}}}{-\frac{1}{x^2}} = \lim_{x\to+\infty}\frac{-4x}{x-4} = -4$$

Outline

Common Mistakes when Using L'Hôpital's Rule

L'Hôpital's rule compute $\frac{0}{0}$ or $\frac{\infty}{\infty}$ type limit, it doesn't solve every limit.

Example (Incorrect Application of L'Hôpital's rule)

Try to use L'Hôpital's rule to evaluate $\lim_{x\to 3} \frac{2x+7}{4x+1}$.

Apply L'Hôpital's rule to the limit and then evaluate.

$$\lim_{x\to 3} \frac{2x+7}{4x+1} = \lim_{x\to 3} \frac{\frac{d}{dx}(2x+7)}{\frac{d}{dx}(4x+1)} = \lim_{x\to 3} \frac{2}{4} = \frac{1}{2}$$

However, this is incorrect! The function near $x = 3$ is continuous and the actual limit value should be 1.

What is wrong with it?

Both $\lim_{x\to 3}(2x+7)$ and $\lim_{x\to 3}(4x+1)$ are not 0 or ∞.

WARNING: L'Hôpital's rule is NOT a universal tool

Example (Failure of L'Hôpital's rule)

Try to use L'Hôpital's rule to evaluate $\lim_{x\to\infty} \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

We first check the condition of L'Hôpital's rule

$$\lim_{x\to\infty}(e^x - e^{-x}) = \infty - 0 = \infty, \quad \lim_{x\to\infty}(e^x + e^{-x}) = \infty + 0 = \infty.$$

Hence it has the form of $\frac{\infty}{\infty}$. We apply L'Hôpital's rule:

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{\frac{d}{dx}(e^x - e^{-x})}{\frac{d}{dx}(e^x + e^{-x})} = \lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{\infty}{\infty}$$

We can easily see that repeated applications of L'Hôpital's rule will just result in the function flipping over and over.

WARNING: L'Hôpital's rule is NOT a universal tool

Example (Failure of L'Hôpital's rule)

It is easy to evaluate $\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$ by simplifying step at first.

We just need to multiply $\frac{e^{-x}}{e^{-x}}$ as follows:

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{(e^x - e^{-x})}{(e^x + e^{-x})} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1-0}{1+0} = 1.$$

WARNING: L'Hôpital's rule is NOT a universal tool

Example (Failure of L'Hôpital's rule)

Try to use L'Hôpital's rule to evaluate $\lim_{x\to\infty} \dfrac{\sqrt{4x^2+3}}{x+3}$.

It is easy to see the limit has $\dfrac{\infty}{\infty}$ form. Using L'Hôpital's rule, we have

$$
\begin{aligned}
\lim_{x\to\infty} \frac{\sqrt{4x^2+3}}{x+3} &= \lim_{x\to\infty} \frac{(4x^2+3)^{1/2}}{x+3} && \text{Rewriting the original limit} \\
&= \lim_{x\to\infty} \frac{\frac{d}{dx}(4x^2+3)^{1/2}}{\frac{d}{dx}(x+3)} && \text{L'Hôpital's rule} \\
&= \lim_{x\to\infty} \frac{\frac{1}{2}(4x^2+3)^{-1/2}\cdot 8x}{1} \\
&= \lim_{x\to\infty} \frac{4x}{(4x^2+3)^{1/2}} = \frac{\infty}{\infty}.
\end{aligned}
$$

WARNING: L'Hôpital's rule is NOT a universal tool

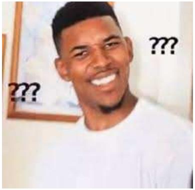
Example (Failure of L'Hôpital's rule)

Apply L'Hôpital's a second time, and re-evaluate the limit.

$$\lim_{x\to\infty}\frac{4x}{(4x^2+3)^{1/2}} = \lim_{x\to\infty}\frac{\frac{d}{dx}(4x)}{\frac{d}{dx}(4x^2+3)^{1/2}} \qquad \text{L'Hôpital's rule}$$

$$= \lim_{x\to\infty}\frac{4}{\frac{1}{2}(4x^2+3)^{-1/2}\cdot 8x}$$

$$= \lim_{x\to\infty}\frac{(4x^2+3)^{1/2}}{x} = \frac{\infty}{\infty} \qquad \text{L'Hôpital's rule again?}$$



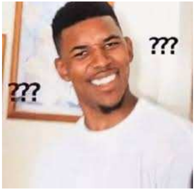
WARNING: L'Hôpital's rule is NOT a universal tool

Example (Failure of L'Hôpital's rule)

$$\cdots = \lim_{x\to\infty} \frac{(4x^2+3)^{1/2}}{x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(4x^2+3)^{1/2}}{\frac{d}{dx}(x)}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{2}(4x^2+3)^{-1/2}\cdot 8x}{1} = \lim_{x\to\infty} \frac{4x}{(4x^2+3)^{1/2}}$$

This is exactly the function we got at the start of last page! No matter how many times we try using L'Hôpital's's rule, it will never yield a limit value. The process will just keep cycling around and around.



WARNING: L'Hôpital's rule is NOT a universal tool

L'Hôpital's rule guarantees that

$$\underbrace{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}_{\text{if the limits exist, the limits are equal.}}$$

However, it does not guarantee the second limit can be evaluated.

Most of the time the second (or third, or fourth, or ...) limits we get from the rule have a simpler form and the new limit is more easily evaluated. But as this example shows, that isn't always the case.

Instead of repeating L'Hôpital's rule, it is important (even critical) to simplify the function at each stage.

WARNING: L'Hôpital's rule is NOT a universal tool

Example (Failure of L'Hôpital's rule)

Factor the x^2 out of the square-root:

$$\lim_{x\to\infty} \frac{\sqrt{4x^2+3}}{x+3} = \lim_{x\to\infty} \frac{\sqrt{x^2\left(4+\frac{3}{x^2}\right)}}{x+3} = \lim_{x\to\infty} \frac{\sqrt{x^2}\cdot\sqrt{4+\frac{3}{x^2}}}{x+3}$$

$$= \lim_{x\to\infty} \frac{|x|\cdot\sqrt{4+\frac{3}{x^2}}}{x+3} = \lim_{x\to\infty} \frac{x\cdot\sqrt{4+\frac{3}{x^2}}}{x+3}$$

Since $x\to\infty$, we replace $|x|$ with just x. Factor the x out of the denominator:

$$\lim_{x\to\infty} \frac{x\cdot\sqrt{4+\frac{3}{x^2}}}{x+3} = \lim_{x\to\infty} \frac{x\cdot\sqrt{4+\frac{3}{x^2}}}{x\left(1+\frac{3}{x}\right)} = \lim_{x\to\infty} \frac{\sqrt{4+\frac{3}{x^2}}}{1+\frac{3}{x}} = \frac{\lim\limits_{x\to\infty}\sqrt{4+\frac{3}{x^2}}}{\lim\limits_{x\to\infty}\left(1+\frac{3}{x}\right)} = 4.$$

L'Hôpital's Rule

1. L'Hôpital's rule help us compute limits of the $\frac{0}{0}$ or $\frac{\infty}{\infty}$-type
2. L'Hôpital's rule is not a universal tool.
3. We must check the form of limit before applying L'Hôpital's rule.
4. Sometimes, simplifying the expression is more useful.

Outline

We study how to find the minimum of a convex function f.

This section is beyond the requirement of MATH 1013. It will NOT been contained in our homework or exam, however, it is very helpful to understand the concepts of convex function, linear approximation, optimization problem and Newton methods (next week).

Convex Optimization

We introduce the following assumptions:

1. the domain of f is $(-\infty, \infty)$
2. f is twice differentiable
3. for any x, we have $f''(x) \leq L$ for some positive constant L
4. there exists x^* such that $f(x^*)$ is the minimum

Note that there could be no closed form expression of x^* or $f(x^*)$.

Hence, we desire to generate the sequence

$$x_0, x_1, x_2, x_3 \ldots$$

such that $|x_k - x^*|$ or $|f(x_k) - f(x^*)|$ convergences to 0 with increasing k.

Exercise

If f is twice differentiable on (a, b) and continuous on $[a, b]$, then

$$f(b) - f(a) - f'(a)(b-a) = \frac{f''(c)}{2}(b-a)^2$$

for some $c \in (a, b)$.

Let $a = x_k$ and $b = x \neq x_k$, then there exists c such that

$$\begin{aligned} f(x) =& f(x_k) + f'(x_k)(x - x_k) + \frac{f''(c)}{2}(x - x_k)^2 \\ \leq& f(x_k) + f'(x_k)(x - x_k) + \frac{L}{2}(x - x_k)^2 \end{aligned}$$

Convex Optimization

The inequality

$$f(x) \leq f(x_k) + f'(x_k)(x - x_k) + \frac{L}{2}(x - x_k)^2$$

provides an upper bound of $f(x)$.

By fixing x_k, the minimizer of the upper bound

$$g_k(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{L}{2}(x - x_k)^2$$

has closed form solution! We have $g_k'(x) = f'(x_k) + L(x - x_k)$, then

$$g_k'(x) = 0 \Longrightarrow f'(x_k) + L(x - x_k) = 0 \Longrightarrow x = x_k - \frac{1}{L}f'(x_k)$$

We can select any x_0 as initial point and run iteration

$$x_{k+1} = x_k - \frac{1}{L}f'(x_k) \quad \text{with } k = 1, 2 \ldots \ldots$$

For any initial point x_0, the iteration scheme

$$x_{k+1} = x_k - \frac{1}{L} f'(x_k) \quad \text{with } k = 1, 2 \ldots \ldots$$

satisfies

$$f(x_k) - f(x^*) \leq \frac{|x_0 - x^*|}{2Lk}.$$

Since $|x_0 - x^*|$ and L are independent to k, we have

$$\lim_{k \to \infty} \frac{|x_0 - x^*|}{2Lk} = 0.$$

Hence, $f(x_k)$ convergences to $f(x^*)$.

Convex Optimization

If we additionally suppose f is strongly convex with constant c.

Then for any initial point x_0, the iteration scheme

$$x_{k+1} = x_k - \frac{1}{L} f'(x_k) \quad \text{with } k = 1, 2 \ldots\ldots$$

satisfies

$$f(x_k) - f(x^*) \le \left(1 - \frac{c}{L}\right)^k (f(x_0) - f(x^*)).$$

In fact, we can prove $c \le L$ which means

$$\lim_{k \to \infty} \left(1 - \frac{c}{L}\right)^k (f(x_0) - f(x^*)) = 0.$$

Hence, $f(x_k)$ convergences to $f(x^*)$.

Convex Optimization

By comparing the convergence result

$$f(x_k) - f(x^*) \leq \frac{|x_0 - x^*|}{2Lk} \quad \text{for convex } f$$

and

$$f(x_k) - f(x^*) \leq \left(1 - \frac{c}{L}\right)^k (f(x_0) - f(x^*)) \quad \text{for strongly convex } f,$$

we have

$$\lim_{k\to\infty} \frac{\left(1 - \frac{c}{L}\right)^k (f(x_0) - f(x^*))}{\frac{|x_0 - x^*|}{2Lk}} = 0.$$

The iteration

$$x_{k+1} = x_k - \frac{1}{L} f'(x_k) \quad \text{with } k = 1, 2 \ldots \ldots$$

is called gradient descent, which is the most popular method to find the minimum of a convex function.

A convex function on a closed interval I must have a maximum on the endpoint of I.

Calculus IB: Lecture 17

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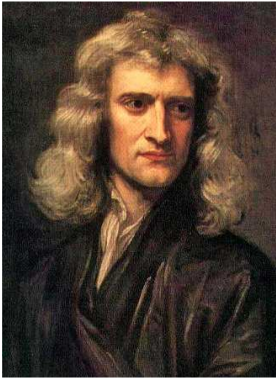
Outline

Outline

1. Newton's Method: Algorithm
2. Newton's Method: Convergence Analysis
3. Newton's Method: Application in Convex Optimization

Newton's Method

Newton's method, also known as the Newton–Raphson method, named after Isaac Newton and Joseph Raphson (who can provide a picture/photo of Raphson?), is a root-finding algorithm.



Sir Isaac Newton (1642-1726), the greatest scientist of all time.

Newton's Method

Newton's method is a simple usage of the tangent lines in finding approximate solutions of a non-linear equation

$$f(x) = 0,$$

where f is differentiable and defined on real numbers.

Note that non-linear equation may have no closed form solution, e.g.

$$f(x) = x^5 - x + 1 = 0.$$

In many optimization problems, one important step is finding critical point, which just corresponds to solving a non-linear equation without closed form solution.

Newton's Method

The Newton's method generates sequence

$$x_0, x_1, x_2, x_3 \dots$$

such that $f(x_k)$ converges to 0 with increasing k.

The basic idea is applying linear approximation on $f(x)$ at given x_k

$$f(x) \approx f(x_k) + f'(x_k)(x - x_k).$$

Then we solve the linear equation

$$f(x_k) + f'(x_k)(x - x_k) = 0$$

is an approximation of solving $f(x) = 0$.

Newton's Method

Let x_{k+1} be the solution of

$$f(x_k) + f'(x_k)(x - x_k) = 0.$$

Note that $y = g_k(x) = f(x_k) + f'(x_k)(x - x_k)$ is a linear function whose slope is $f'(x_k)$ and graph passes the point $(x_k, f(x_k))$

Suppose $f'(x_k) \neq 0$, then we have

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{1}$$

Newton's method iterates (1) from a suitable initial point x_0.

Roughly speaking, if initial point x_0 is not too far away from a root of $f(x) = 0$, Newton's method produces a sequence $x_1, x_2, x_3, \ldots$, which may get closer and closer to an exact root of $f(x) = 0$.

Example

Find an approximate positive root of $x^3 - 2 = 0$ by the Newton's method. (exact root $\sqrt[3]{2} = 1.25992104989\cdots$ which can be obtained by calculator)

Let $f(x) = x^3 - 2$, and hence $f'(x) = 3x^2$ and the iteration formula is

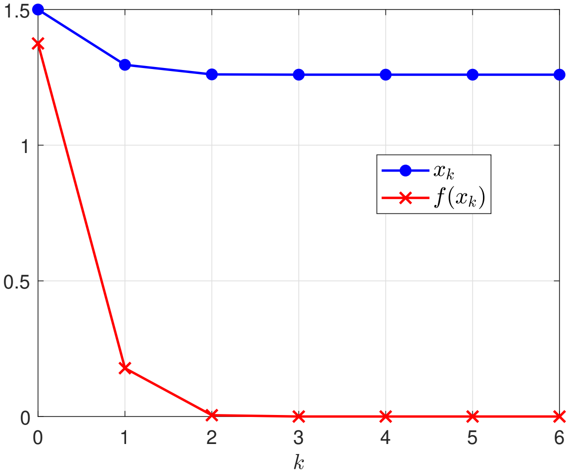
$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} = x_k - \frac{x_k^3 - 2}{3x_k^2}$$

k	x_k	$f(x_k) = x_k^3 - 2$	$f'(x_k) = 3x_k^2$	$x_{k+1} = x_k - \frac{x_k^3-2}{3x_k^2}$
0	1.500000000	1.375000000	6.750000000	1.296296296
1	1.296296296	0.178275669	5.041152263	1.260932225
2	1.260932225	0.004819286	4.769850226	1.259921861
3	1.259921861	0.000003861	4.762209284	1.259921050
4	1.259921050	0.000000000	4.762203156	1.259921050
5	1.259921050	0.000000000	4.762203156	1.259921050
6	1.259921050	0.000000000	4.762203156	1.259921050

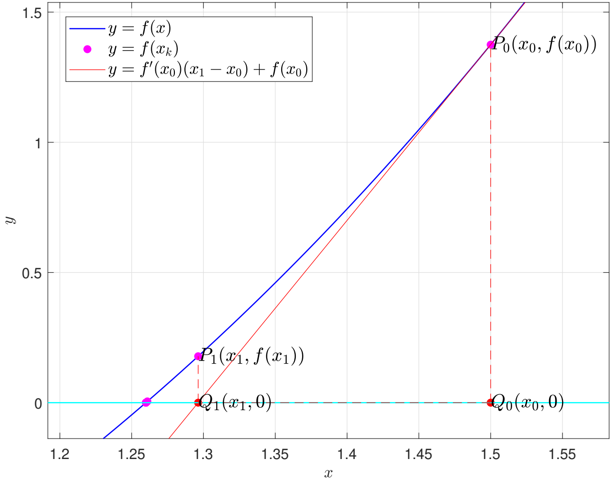
An approximate value of $\sqrt[3]{2}$ is 1.259921050.

Newton's Method

Convergence behavior of x_k and $f(x_k)$ with iterations:



Newton's Method



$$x_0 - x_1 = Q_1 Q_0 = \frac{P_0 Q_0}{\tan \angle P_0 Q_1 Q_0} = \frac{f(x_0)}{f'(x_0)} \Longrightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Newton's Method

Exercise

Find an approximate value of the root of the equation

$$\cos x - x = 0$$

by the Newton's Method (using calculator or MATLAB).

Outline

1. Newton's Method: Algorithm
2. Newton's Method: Convergence Analysis
3. Newton's Method: Application in Convex Optimization

Convergence Analysis

Above example show that Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

produces a sequence

$$x_1, x_2, x_3, \ldots$$

which converge to the solution x^* such that $f(x^*) = 0$.

We impose the following assumptions to further analysis

1. the function f is differentiable
2. there exists positive μ such that $|f'(x)| \geq \mu$ for all x
3. there exists positive C such that $|f''(x)| \leq C$ for all x
4. the initial point is NOT far away form x^*

(We typically suppose $f'(x)$ is C-Lipschitz continuous, rather than $|f''(x)| \leq C$, but the related analysis needs some techniques beyond MATH 1013.)

Mean Value Theorem

If f is twice differentiable on (a, b) and continuous on $[a, b]$, then

$$f(b) - f(a) - f'(a)(b - a) = \frac{f''(c)}{2}(b - a)^2$$

for some $c \in (a, b)$. If $a > b$ and function f is twice differentiable on (b, a) and continuous on $[b, a]$, we can also find c between a and b satisfies above inequality.

The mean value theorem means there exists c_k between x_k and x^* that

$$f(x^*) - f(x_k) - f'(x_k)(x^* - x_k) = \frac{f''(c_k)}{2}(x^* - x_k)^2.$$

Convergence Analysis

Using the iteration of Newton's Method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}.$$

and the result from mean value theorem

$$f(x^*) - f(x_k) - f'(x_k)(x^* - x_k) = \frac{f''(c_k)}{2}(x^* - x_k)^2,$$

we have

$$\begin{aligned}
|x_{k+1} - x^*| &= \left| x_k - \frac{f(x_k)}{f'(x_k)} - x^* \right| \\
&= \frac{1}{|f'(x_k)|} \left| f'(x_k) \cdot (x_k - x^*) - (f(x_k) - f(x^*)) \right| \\
&= \frac{1}{|f'(x_k)|} \left| \frac{f''(c_k)}{2}(x^* - x_k)^2 \right| = \frac{|f''(c_k)|}{2|f'(x_k)|} |x_k - x^*|^2
\end{aligned}$$

Convergence Analysis

Since $|f'(x)| \geq \mu$ and $|f''(x)| \leq C$, we have

$$|x_{k+1} - x^*| = \frac{|f''(c_k)|}{2|f'(x_k)|}|x_k - x^*|^2 \leq \frac{C}{2\mu}|x_k - x^*|^2.$$

Then we have (try to prove the last one by induction)

$$\begin{cases} |x_1 - x^*| \leq \frac{C}{2\mu}|x_0 - x^*|^2 \\ |x_2 - x^*| \leq \frac{C}{2\mu}|x_1 - x^*|^2 \leq \left(\frac{C}{2\mu}\right)^3 |x_0 - x^*|^4 \\ |x_3 - x^*| \leq \frac{C}{2\mu}|x_2 - x^*|^2 \leq \left(\frac{C}{2\mu}\right)^7 |x_0 - x^*|^8 \\ \qquad \cdots\cdots \\ |x_k - x^*| \leq \frac{C}{2\mu}|x_{k-1} - x^*| \leq \left(\frac{C}{2\mu}\right)^{2^k-1} |x_0 - x^*|^{2^k} \end{cases}$$

Convergence Analysis

In summary Newton's method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

from x_0 holds that

$$|x_k - x^*| \leq \left(\frac{C}{2\mu}\right)^{2^k-1} |x_0 - x^*|^{2^k} \leq \frac{2\mu}{C}\left(\frac{C}{2\mu}|x_0 - x^*|\right)^{2^k}.$$

If $\dfrac{C}{2\mu}|x_0 - x^*| < 1$, the sequence will $x_1, \ldots, x_k$ converges to x^* very fast.

Otherwise, it may diverge. Hence, we should start with x_0 such that

$$|x_0 - x^*| < \frac{2\mu}{C}.$$

Outline

Newton's Method in Convex Optimization

The basic idea is using Newton's method to solve $f'(x) = 0$, since the equation has no closed form solution in many situations.

Theorem

If function f is convex and differentiable over an interval I. Then any point x^* that satisfies $f'(x^*) = 0$ holds that $f(x^*)$ is a global minimum.

We consider the simple case that $I = (-\infty, \infty)$.

Newton's Method in Convex Optimization

Impose previous assumption on f':

1. the function f' differentiable
2. there exists positive μ such that $|f''(x)| \geq \mu$ for all x
3. there exists positive C such that $|f'''(x)| \leq C$ for all x
4. the initial point x_0 satisfies $|x_0 - x^*| < \frac{2\mu}{C}$

In the view of convex optimization:

1. the function f is twice differentiable
2. the function f is strongly convex with factor $\mu > 0$, i.e. $f''(x) \geq \mu$.
3. there exists positive C such that $|f'''(x)| \leq C$ for all x
4. the initial point x_0 satisfies $|x_0 - x^*| < \frac{2\mu}{C}$

The third one typically can be replaced by "there exists $C > 0$ such that for any x_1 and x_2, we have $|f''(x_1) - f''(x_2)| \leq C|x_1 - x_2|$".

According to the analysis of Newton's method, we have

$$|x_k - x^*| \leq \frac{2\mu}{C}\left(\frac{C}{2\mu}|x_0 - x^*|\right)^{2^k}.$$

By assuming $f''(x) \leq L$, we can establish the result of function value

$$f(x_k) - f(x^*) - f'(x^*)(x_k - x^*) = \frac{f''(c_k)}{2}(x^* - x_k)^2$$

$$\Longrightarrow f(x_k) - f(x^*) \leq \frac{L}{2}(x^* - x_k)^2 \leq \frac{L}{2}\left(\frac{2\mu}{C}\right)^2\left(\frac{C}{2\mu}|x_0 - x^*|\right)^{2^{k+1}}$$

$$\Longrightarrow f(x_k) - f(x^*) \leq \frac{2\mu^2 L}{C^2}\left(\frac{C}{2\mu}|x_0 - x^*|\right)^{2^{k+1}}$$

Newton's Method in Convex Optimization

According to the analysis of Newton's method, we have

$$f(x_k) - f(x^*) \leq \frac{2\mu^2 L}{C^2}\left(\frac{C}{2\mu}|x_0 - x^*|\right)^{2^{k+1}}.$$

If we desire $f(x_k) \approx f(x^*)$ such that $f(x_k) - f(x^*) \leq \varepsilon$, which requires

$$\begin{aligned}
&\frac{2\mu^2 L}{C^2}\left(\frac{C}{2\mu}|x_0 - x^*|\right)^{2^{k+1}} \leq \varepsilon \\
\Longrightarrow &\left(\frac{C}{2\mu}|x_0 - x^*|\right)^{2^{k+1}} \leq \frac{C^2\varepsilon}{2\mu^2 L} \\
\Longrightarrow &2^{k+1} \geq \log_{\frac{C}{2\mu}|x_0 - x^*|}\left(\frac{C^2\varepsilon}{2\mu^2 L}\right) \\
\Longrightarrow &k \geq \log_2\left(\log_{\frac{C}{2\mu}|x_0 - x^*|}\left(\frac{C^2\varepsilon}{2\mu^2 L}\right)\right) - 1.
\end{aligned}$$

Strengths of Newton's Method

Note that

$$k \geq \log_2 \left(\log_{\frac{C}{2\mu}|x_0 - x^*|} \left(\frac{C^2 \varepsilon}{2\mu^2 L} \right) \right) - 1.$$

means we only need very few even if ε is very small.

Since $\dfrac{C}{2\mu}|x_0 - x^*| < 1$, we suppose $\dfrac{C}{2\mu}|x_0 - x^*| = 0.999 \approx 1$.

We also assume ε is very small such that $\dfrac{C^2 \varepsilon}{2\mu^2 L} = 10^{-8}$.

Then we only require $k \geq 13.1683$.

Strengths of Newton's Method

Recall that gradient descent holds that

$$f(x_k) - f(x^*) \leq \left(1 - \frac{\mu}{L}\right)^k (f(x_0) - f(x^*))$$

for strongly convex and differentiable f.

To find x_k such that $f(x_k) - f(x^*) \leq \varepsilon$, it needs

$$\begin{aligned}
&\left(1 - \frac{\mu}{L}\right)^k (f(x_0) - f(x^*)) \leq \varepsilon \\
\Longrightarrow &\left(1 - \frac{\mu}{L}\right)^k \leq \frac{f(x_0) - f(x^*)}{\varepsilon} \\
\Longrightarrow &k \geq \log_{\left(1 - \frac{\mu}{L}\right)} \left(\frac{f(x_0) - f(x^*)}{\varepsilon}\right)
\end{aligned}$$

Strengths of Newton's Method

If it is desired a very very accuracy approximation, we only interested in how ε affects k since ε is much smaller than other terms.

Then gradient descent needs

$$k \geq \log_{C_1}(C_2\varepsilon)$$

and Newton's method needs

$$k \geq \log_2\left(\log_{C_3}(C_4\varepsilon)\right).$$

We have (try to show that by L'Hôpital's rule as an exercise)

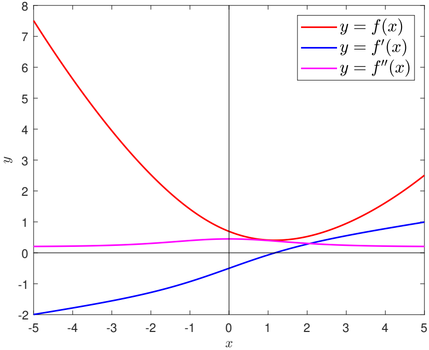
$$\lim_{\varepsilon\to 0^+} \frac{\log_2\left(\log_{C_3}(C_4\varepsilon)\right)}{\log_{C_1}(C_2\varepsilon)} = 0.$$

Strengths of Newton's Method

Consider the example of finding minimum of

$$f(x) = \frac{x^2}{10} + \ln(1 + e^{-x}),$$

we have $f'(x) = \dfrac{x}{5} - \dfrac{e^{-x}}{e^{-x} + 1}$ and $f''(x) = \dfrac{e^{2x} + 7e^2 + 1}{5(e^x + 1)^2} \leq \dfrac{9}{2}$.



Strengths of Newton's Method

We run gradient descent

$$x_{k+1} = x_k - \frac{1}{L} f'(x_k)$$

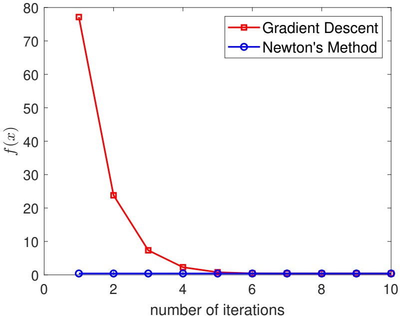
with $L = \dfrac{9}{2}$ and Newton's method

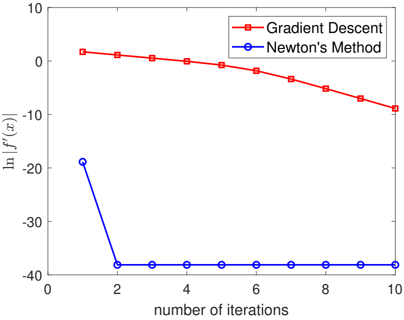
$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}.$$

We select $x_0 = 10$ and run 10 iterations for both algorithms.

Strengths of Newton's Method

The convergence behavior of $f(x)$ and $\ln|f'(x)|$. In theoretical, $|f'(x)|$ should tend to 0 which leads to $\ln|f'(x)| \to -\infty$, but the computer cannot present too small magnitude of a real number if it is not 0.





Weakness of Newton's Method

The initial point x_0 of Newton's method should be near x^*:

$$\frac{C}{2\mu}|x_0 - x^*| < 1.$$

Unfortunately, there is no good strategy to select x_0 for general f since we do not know what is x^* at first.

In other words, the convergence of Newton's method is local, not global.

On the other hand, gradient descent could converge to the optimal solution with any initial point x_0 (global convergence).

Weakness of Newton's Method

Newton's method depends on the twice differentiability while gradient descent only requires first differentiability.

Newton's method works only if $f''(x_k) \neq 0$ which is unnecessary for gradient descent.

If f has many input variables, the computational cost of Newton's method is much more expensive than gradient descent.

Weakness of Newton's Method

Consider using Newton's method to solve

$$f(x) = x^2 - 5 = 0.$$

If we let $x_0 = 0$, then $f'(x_0) = 2x_0 = 0$ and the update

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \text{is undefined.}$$

Weakness of Newton's Method

Newton's method may fail even if $f'(x_k) \neq 0$ for any k ~~when x_0 is far way from the solution~~ and x_0 is very close to x^*. Note that that f' may be undefined at x^*.

Exercise

Consider solving

$$f(x) = x^{\frac{1}{3}} = 0$$

by Newton's method with initial point $x_0 = 1$.

Exercise

Compare the convergence of Newton's method with bisection method (Lecture 08). Which one is faster?

Calculus IB: Lecture 18

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Outline

Outline

Antiderivatives

1. Differentiation problem: Given a function $f \longrightarrow$ find $\dfrac{df}{dx}$.
2. Reversing the process: Given a function $f \longrightarrow$ find a function F such that $F' = f$.
3. This can also be considered as a question of solving the "differential equation"
$$\frac{d}{dx}(\text{which function}) = f(x)\ .$$
4. Any function F satisfying $F' = f$ is called an *antiderivative* (or a primitive function) of f.

Antiderivatives

1. Any function F satisfying $F' = f$ is called an *antiderivative* (or a primitive function) of f.
2. Obviously, if F is an antiderivative of f, then so is $F + C$ for any constant C, since $\dfrac{dC}{dx} = 0$.
3. Note that if F and G are two antiderivatives of f on an open interval, then we have

$$(F - G)' = F' - G' = f - f = 0 \ .$$

   By the mean value theorem, $F - G$ must then be a constant function on the interval; i.e., $G(x) - F(x) = C$ for some constant C.

Antiderivatives

Theorem (Mean Value Theorem)

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some $c \in (a, b)$, or equivalently $f(b) - f(a) = f'(c)(b - a)$.

Let $H(x) = F(x) - G(x)$ defined on $I = (p, q)$ and $H'(x) = 0$ for all x in I. Then for any $p < a < b < q$, we have $f(b) - f(a) = f'(c)(b - a) = 0$ for some $c \in (a, b)$.

Therefore if one antiderivative F has been found for a given function f on an open interval, all antiderivatives of f on the interval can be expressed in the form $F + C$, where C is an arbitrary constant.

Antiderivatives

Example

Let $f(x) = 3x^2$. Solve the antiderivative problem: $\dfrac{d}{dx}(?) = 3x^2$.

Knowing that

$$\frac{dx^3}{dx} = 3x^2,$$

the antiderivatives of $3x^2$ are given by $x^3 + C$, where C is an arbitrary constant.

Antiderivatives

Example

Let $g(x) = 2\cos x$. Solve the antiderivative problem: $\frac{d}{dx}(?) = 2\cos x$.

Since

$$\frac{d\sin x}{dx} = \cos x,$$

it easy to see that

$$\frac{d(2\sin x)}{dx} = 2\cos x.$$

Hence the antiderivatives of $2\cos x$ are given by $2\sin x + C$, where C is an arbitrary constant.

Example

Let $h(x) = x + e^{2x}$. Solve the antiderivative problem: $\dfrac{d}{dx}(?) = x + e^{2x}$

Since

$$\frac{dx^2}{dx} = 2x \quad \text{and} \quad \frac{de^{2x}}{dx} = 2e^{2x},$$

we have

$$\frac{d}{dx}\left(\frac{1}{2}x^2 + \frac{1}{2}e^{2x}\right) = x + e^{2x}.$$

Hence, the antiderivatives of $h(x) = x + e^{2x}$ are given by

$$\left(\frac{1}{2}x^2 + \frac{1}{2}e^{2x}\right) + C.$$

Indefinite Integral

The *indefinite integral* notation

$$\int f(x)dx$$

is nothing but a new dress of the antiderivatives! The function $f(x)$ appearing in an indefinite integral is usually called the *integrand*.

Indefinite Integral

For example,

$$\int 3x^2 dx \quad \overset{means}{=} \text{all antiderivatives of } 3x^2$$

$$\overset{thus}{=} x^3 + C \qquad \left(\text{since } \frac{dx^3}{dx} = 3x^2\right)$$

Equivalently, this is the same as saying that the general solution of the differential equation

$$\frac{dy}{dx} = 3x^2$$

is

$$y = x^3 + C.$$

Example

Show that $\int (2x+1)e^{x^2+x}dx = e^{x^2+x} + C$.

This is just another way to say

$$\frac{d}{dx}(e^{x^2+x}) = e^{x^2+x} \cdot \frac{d(x^2+x)}{dx} = (2x+1)e^{x^2+x}.$$

$(2x+1)e^{x^2+x}$ is the derivative of e^{x^2+x}	$\overset{vs}{\Longleftrightarrow}$	e^{x^2+x} is an antiderivative of $(2x+1)e^{x^2+x}$

In fact, we have

$$\int f(x)dx = F(x) + C \iff \frac{dF}{dx} = f(x).$$

In particular,

$$\frac{d}{dx}\int f(x)dx = f(x),$$

and

$$\int f'(x)dx = f(x) + C.$$

Indefinite Integral

$$\frac{d}{dx}\frac{1}{p+1}x^{p+1} = x^p \quad \overset{p\neq-1}{\Longleftrightarrow} \quad \int x^p dx = \frac{1}{p+1}x^{p+1} + C$$

$$\frac{d}{dx}e^x = e^x \quad \Longleftrightarrow \quad \int e^x dx = e^x + C$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \quad \Longleftrightarrow \quad \int \frac{1}{x} dx = \ln|x| + C$$

$$\frac{d}{dx}\sin x = \cos x \quad \Longleftrightarrow \quad \int \cos x dx = \sin x + C$$

$$\frac{d}{dx}[-\cos x] = \sin x \quad \Longleftrightarrow \quad \int \sin x dx = -\cos x + C$$

$$\frac{d}{dx}\tan x = \sec^2 x \quad \Longleftrightarrow \quad \int \sec^2 x dx = \tan x + C$$

$$\vdots$$

Note that
$$\int \frac{1}{x}dx \neq \ln x + C,$$
since the domain of $\ln x$ is $(0, \infty)$, rather than all real numbers.

Exercise

Check the formula
$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$
and
$$\int \frac{1}{x}dx = \ln|x| + C.$$

Indefinite Integral

The term-by-term differentiation rule

$$(aF(x) + bG(x))' = aF'(x) + bG'(x),$$

where a, b are constants, can be rewritten as an integration rule:

$$\int (af(x) + bg(x))dx = a\int f(x)dx + b\int g(x)dx$$

Just note that derivatives of both sides are equal to

$$af(x) + bg(x);$$

i.e., both sides are antiderivatives of

$$af(x) + bg(x).$$

Indefinite Integral

Example

1. $$\int \left(3x^4 - 2x^3\right) dx = 3\int x^4 dx - 2\int x^3 dx = \frac{3}{5}x^5 - \frac{1}{2}x^4 + C$$
2. $$\int \left(x^5 - e^x + \frac{1}{x}\right) dx = \int x^5 dx - \int e^x dx + \int \frac{1}{x} dx$$
   $$= \frac{1}{6}x^6 - e^x + \ln|x| + C$$
3. $$\int \left(3\sin x - 2\sec^2 x + 3e^x\right) dx = -3\cos x - 2\tan x + 3e^x + C$$

Outline

The Substitution Rule

So far, most of our integrals could be found directly, up to some algebraic manipulations.

When an integral looks complicated, without an obvious antiderivative, quite often we need to make it simpler by using a suitable "substitution".

The Substitution Rule

If you know

$$\frac{d}{dx}\sin x^2 = 2x\cos x^2,$$

then it is of course straightforward to write

$$\int 2x\cos x^2 dx = \sin x^2 + C.$$

What if you don't know the derivative?

Let $u = x^2$, so that

$$\frac{du}{dx} = 2x.$$

Now, by formally writing $du = 2xdx$ and putting everything into the original integral, we have

$$\int {\color{red}2x}\cos x^2 {\color{red}dx} = \int \cos u\,{\color{red}du} \overset{easy}{=} \sin u + C = \sin x^2 + C$$

The Substitution Rule

By letting $u = g(x)$ (some way to group some perhaps complicated x expression as u), and $du = g'(x)dx$, the following may happen:

$$\text{Apply } u = g(x) \text{ and } du = g'(x)dx \text{ on } \int f(x)dx$$

$$\Longrightarrow \text{an easier } u\text{-integral } \int F(u)du$$

$$\Longrightarrow \text{put } u = g(x) \text{ back after finishing the } u\text{-integral}$$

The Substitution Rule

The reason behind the Substitution Rule is Chain Rule!

The chain rule means

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x).$$

By letting $u = g(x)$, $du = g'(x)dx$, we have

$$\int f(g(x))g'(x)dx = \int f(u)du = F(u) + C.$$

Then $F'(u) = f(u)$ and

$$\frac{d}{dx}F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)dx$$

and hence the antiderivatives of $f(g(x))g'(x)$ are given by

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

The Substitution Rule

Theorem

If $u = g(x)$ is a differentiable function whose range is an interval I, and $f(x)$ is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Example

Find the indefinite integral of $\int \sqrt{3x+2}dx$.

Let $u = 3x + 2$ such that

$$\frac{du}{dx} = 3 \quad \text{and} \quad \frac{1}{3}du = dx.$$

Hence

$$\begin{aligned}\int \sqrt{3x+2}dx =& \int \frac{1}{3}u^{1/2}du \\ =& \frac{1}{3} \cdot \frac{2}{3}u^{3/2} + C \\ =& \frac{2}{9}(3x+2)^{3/2} + C\end{aligned}$$

The Substitution Rule

Example

Find the indefinite integral of $\displaystyle\int \sin(5x+2)dx$.

Let $u = 5x + 2$ such that

$$\frac{du}{dx} = 5 \quad \text{and} \quad \frac{1}{5}du = dx.$$

Hence

$$\begin{aligned}\int \sin(5x+2)dx &= \int \frac{1}{5}\sin u du \\ &= -\frac{1}{5}\cos u + C \\ &= -\frac{1}{5}\cos(5x+2) + C\end{aligned}$$

The Substitution Rule

Example

Find the indefinite integral of $\int \frac{1}{2x+1} dx$.

Let $u = 2x + 1$ such that

$$\frac{du}{dx} = 2 \quad \text{and} \quad \frac{1}{2} du = dx.$$

Hence

$$\begin{aligned}\int \frac{1}{2x+1} dx =& \frac{1}{2} \int \frac{1}{u} du \\ =& \frac{1}{2} \ln |u| + C \\ =& \frac{1}{2} \ln |2x+1| + C\end{aligned}$$

Example

Find the indefinite integral of $\displaystyle\int x^2 e^{x^3+1} dx$.

Let $u = x^3 + 1$ such that

$$\frac{du}{dx} = 3x^2 \quad \text{and} \quad \frac{1}{3} du = x^2 dx.$$

Hence

$$\begin{aligned}\int x^2 e^{x^3+1} dx =& \frac{1}{3} \int e^u du \\ =& \frac{1}{3} e^u + C \\ =& \frac{1}{3} e^{x^3+1} + C\end{aligned}$$

The Substitution Rule

Example

Find the indefinite integral of $\int \frac{3t}{4t^2 - 1} dt$.

Let $u = 4t^2 - 1$ such that

$$\frac{du}{dt} = 8t \quad \text{and} \quad \frac{1}{8} du = t dt.$$

Hence

$$\begin{aligned} \int \frac{3t}{4t^2 - 1} dt =& \frac{3}{8} \int \frac{1}{u} du \\ =& \frac{3}{8} \ln |u| + C \\ =& \frac{3}{8} \ln |4t^2 - 1| + C \end{aligned}$$

The Substitution Rule

Exercise

Find the following indefinite integral

1. $\displaystyle\int \frac{x}{\sqrt{x^2+1}}dx$
2. $\displaystyle\int \sec x dx$ (hint: let $u = \sec x + \tan x$)
3. $\displaystyle\int \sin^3\theta d\theta$
4. $\displaystyle\int \cos^3\theta d\theta$
5. $\displaystyle\int xe^{6x^2}dx$

Outline

1. Antiderivatives/Indefinite Integral
2. The Substitution Rule
3. Integration by Parts

Let's study the integral

$$\int x e^{6x} dx.$$

If the integrand was e^{6x^2}, we could do the integral with a substitution $u = x^2$. Unfortunately, such idea does not work here.

To do this integral we will need to use integration by parts.

Integration by Parts

We'll start with the product rule of derivative

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

Now integrate both sides of this formula

$$\begin{aligned} f(x)g(x) &= \int [f(x)g(x)]'\, dx \\ &= \int \left[f'(x)g(x) + f(x)g'(x)\right] dx \\ &= \int f'(x)g(x)dx + \int f(x)g'(x)dx \end{aligned}$$

Substituting $u = f(x)$, $v = g(x)$, $du = f'(x)dx$ and $dv = g'(x)dx$, then

$$\int udv = uv - \int vdu$$

Integration by Parts

The following formula is called integration by parts

$$\int u dv = uv - \int v du$$

To use this formula, we will need to identify u and dv, then compute

$$\int v du$$

Example

Find the indefinite integral of $\displaystyle\int xe^{6x}dx$.

Let $u = x$ and $dv = e^{6x}dx$, then $du = dx$ and

$$v = \int e^{6x}dx = \frac{1}{6}e^{6x}$$

Using integration by parts, we have

$$\begin{aligned}\int xe^{6x}dx &= \int udv = uv - \int vdu \\ &= \frac{x}{6}e^{6x} - \int \frac{1}{6}e^{6x}dx = \frac{x}{6}e^{6x} - \frac{1}{36}e^{6x} + C\end{aligned}$$

Integration by Parts

Example

Find the indefinite integral of $\int (3t+5)\cos\left(\frac{t}{4}\right) dt$.

Let $u = 3t+5, dv = \cos\left(\frac{t}{4}\right) dt, du = 3dt, v = 4\sin\left(\frac{t}{4}\right)$.

Using integration by parts, we have

$$\int (3t+5)\cos\left(\frac{t}{4}\right) dt = \int udv = uv - \int vdu$$

$$=4(3t+5)\sin\left(\frac{t}{4}\right) - \int 4\sin\left(\frac{t}{4}\right)(3dt)$$

$$=4(3t+5)\sin\left(\frac{t}{4}\right) - 12\int \sin\left(\frac{t}{4}\right) dt$$

$$=4(3t+5)\sin\left(\frac{t}{4}\right) + 48\cos\left(\frac{t}{4}\right) + C$$

Exercise

Find the following indefinite integral

1. $\displaystyle\int x\sqrt{x+1}dx$

2. $\displaystyle\int \ln x dx$

Calculus IB: Lecture 19

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Outline

Outline

1. Initial Value Problems
2. Area under Curve
3. Riemann Sums and Definite Integrals

Indefinite Integral

$$\frac{d}{dx}\frac{1}{p+1}x^{p+1} = x^p \quad \overset{p\neq -1}{\Longleftrightarrow} \quad \int x^p dx = \frac{1}{p+1}x^{p+1} + C$$

$$\frac{d}{dx}e^x = e^x \quad \Longleftrightarrow \quad \int e^x dx = e^x + C$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \quad \Longleftrightarrow \quad \int \frac{1}{x}dx = \ln|x| + C$$

$$\frac{d}{dx}\sin x = \cos x \quad \Longleftrightarrow \quad \int \cos x dx = \sin x + C$$

$$\frac{d}{dx}[-\cos x] = \sin x \quad \Longleftrightarrow \quad \int \sin x dx = -\cos x + C$$

$$\frac{d}{dx}\tan x = \sec^2 x \quad \Longleftrightarrow \quad \int \sec^2 x dx = \tan x + C$$

$$\vdots$$

Initial Value Problems

The constant C appearing in

$$\int f(x)dx = F(x) + C$$

may be determined uniquely if further condition is imposed on the value of the antiderivative at a specific x_0.

Such a value of the antiderivative is usually called an initial value.

No simple result can summarize which types of further condition could lead to uniqueness. This topic is contained in other course, e.g "Ordinary Differential Equations".

Example

Suppose that the graph of $y = y(x)$ defines a curve passing the point $(1, -2)$, with its slope satisfying $y' = x^2$. Find the function y.

We first find the indefinite integral

$$y = \int x^2 dx = \frac{1}{2+1}x^{2+1} + C = \frac{1}{3}x^3 + C.$$

Consider that $x = 1$, $y = -2$, hence

$$-2 = \frac{1}{3}(1)^3 + C \iff C = -2 - \frac{1}{3} = -\frac{7}{3}$$

i.e., $y = \dfrac{1}{3}x^3 - \dfrac{7}{3}$.

Initial Value Problems

Example

The acceleration of a falling particle near the surface of the earth is approximately $g = 9.8\text{m}/\text{s}^2$. If $v(t)$ is the velocity of the particle, and the initial value at $t = 0$ is $v(0) = v_0$, find $v(t)$.

The initial value problem is: $\dfrac{dv}{dt} = -g$, and $v(0) = v_0$. We have

$$v(t) = -\int g dt = -gt + C.$$

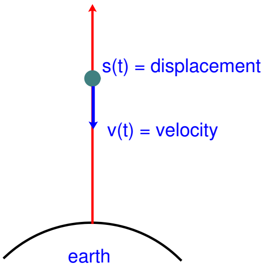
Putting in $t = 0$, we have $v_0 = -9.8(0) + C = C$ i.e., $v(t) = -gt + v_0$.

Initial Value Problems

If $s(t)$ is the displacement function in above example, and the initial position is given as $s(0) = s_0$, then $\frac{ds}{dt} = v(t)$, and

$$s(t) = \int v(t)dt = \int (-gt + v_0)dt = -\frac{1}{2}gt^2 + v_0 t + C$$

Putting in $s(0) = s_0$, we have $C = s_0$ and $s(t) = -\frac{1}{2}gt^2 + v_0 t + s_0$.



Example

The acceleration of a particle moving along a line is given by $a = 2t + 1$. If the initial position of the particle is $s(0) = 4$ and initial velocity $v(0) = -2$, find the position function of the particle. (all quantities are in SI units)

Note that $\dfrac{dv}{dt} = a = 2t + 1$, hence

$$v(t) = \int (2t+1)dt = t^2 + t + C$$

Putting in $t = 0$, we have

$$-2 = v(0) = (0)^2 - 0 + C \iff C = -2$$

i.e., $v(t) = t^2 + t - 2$. Since $s'(t) = v(t)$, we have

$$s(t) = \int (t^2 + t - 2)dt = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + C_1.$$

Putting in $t = 0$, we have $4 = 0 + C_1 = C_1$ and $s(t) = \dfrac{1}{3}t^3 + \dfrac{1}{2}t^2 - 2t + 4$

Outline

1. Initial Value Problems

2. Area under Curve

3. Riemann Sums and Definite Integrals

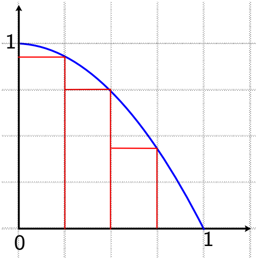
Area Under a Graph By Squeezing

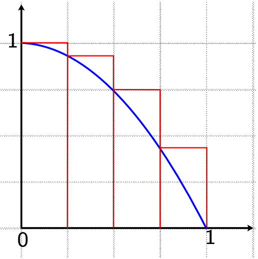
The idea of definite integral is basically from a combination of the approximation of area by rectangles and limit taking.

Let's start with a simple example to illustrate how the area under the graph of a function can be squeezed out by using rectangular approximations of the region.

Problem: Finding the area A under the graph of $y = 1 - x^2$ over the interval $[0, 1]$.

Although it is not immediately clear what the area A is, it is extremely easy to estimate the area A roughly by using just a few rectangles.



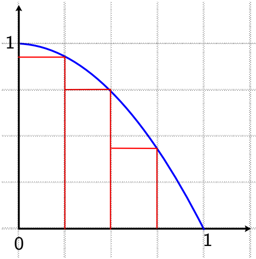


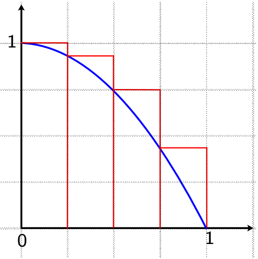
Area Under $y = 1 - x^2$ over $[0, 1]$

1. Divide the interval $[0, 1]$ into 4 subintervals of the same length.
2. Use rectangles based on these subintervals, with heights equal to functions values at the left or right endpoints of the subintervals, to estimate the area.
3. The area A is then squeezed between sums of rectangular areas:

$$A > (1 - (0.25)^2) \cdot 0.25 + (1 - (0.5)^2) \cdot 0.25 + (1 - (0.75)^2) \cdot 0.25 = 0.53125$$

$$\begin{aligned} A < &(1 - 0^2) \cdot 0.25 + (1 - (0.25)^2) \cdot 0.25 \\ &+ (1 - (0.5)^2) \cdot 0.25 + (1 - (0.75)^2) \cdot 0.25 = 0.78125 \end{aligned}$$





Area Under $y = 1 - x^2$ over $[0, 1]$

By subdividing $[0, 1]$ into more and more subintervals, we expect better and better estimates of the area A, and eventually squeezing out the area by taking limit.

More precisely, we start by subdividing the interval $[0, 1]$ into n subintervals of the same length $\frac{1}{n}$ by the subdivision points

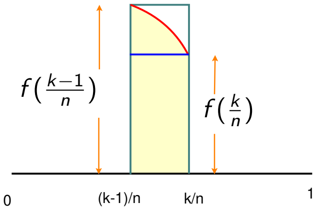
$$0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n}{n} = 1$$

Over the interval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$, $k = 1, \ldots, n$, we have a rectangular area sandwich:

$$\left[1 - \left(\frac{k}{n}\right)^2\right]\frac{1}{n} < A_k < \left[1 - \left(\frac{k-1}{n}\right)^2\right]\frac{1}{n}$$

$$\frac{1}{n} - \frac{k^2}{n^3} < A_k < \frac{1}{n} - \frac{(k-1)^2}{n^3}$$

$A_k =$ shaded area under the graph



Area Under $y = 1 - x^2$ over $[0, 1]$

We can bound the area of k-th parts as follows:

$$\frac{1}{n} - \frac{k^2}{n^3} < A_k < \frac{1}{n} - \frac{(k-1)^2}{n^3}.$$

Adding all these rectangular area sandwiches together, we have

$$\underbrace{n \cdot \frac{1}{n} - \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right)}_{\text{under estimate}(n)} < A < \underbrace{n \cdot \frac{1}{n} - \frac{1}{n^3}\left(0^2 + 1^2 + \cdots + (n-1)^2\right)}_{\text{upper estimate}(n)}.$$

Note that difference of two estimates tends to zero

$$\text{upper estimate}(n) - \text{ lower estimate}(n) = \frac{1}{n} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Area Under $y = 1 - x^2$ over $[0, 1]$

Sandwich theorem means

$$A = \lim_{n\to\infty} \text{upper estimate}(n) = \lim_{n\to\infty} \text{lower estimate}(n).$$

Squeeze Theorem (or Sandwich Theorem)

Let I be an interval having the point a. Let g, f, and h be functions defined on I, except possibly at a itself. Suppose that for every x in I NOT equal to a, we have If $g(x) \leq f(x) \leq h(x)$ for all x near a, except perhaps when $x = a$, then

$$\lim_{x\to a} g(x) \leq \lim_{x\to a} f(x) \leq \lim_{x\to a} h(x)$$

whenever these limits exist. (we allow a be ∞ or $-\infty$)

Area Under $y = 1 - x^2$ over $[0, 1]$

Hence, to find the are under curve, we only needs to find limits

$$\lim_{n\to\infty}\left[n\cdot\frac{1}{n}-\frac{1}{n^3}\left(1^2+2^2+\cdots+n^2\right)\right]=\lim_{n\to\infty}\left[1-\frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}\right]$$

or

$$\lim_{n\to\infty}\left[n\cdot\frac{1}{n}-\frac{1}{n^3}\left(0^2+1^2+\cdots+(n-1)^2\right)\right]=\lim_{n\to\infty}\left[1-\frac{1}{n^3}\frac{(n-1)n(2n-1)}{6}\right].$$

Tutorial/Exercise

Show that

$$1^2+2^2+3^2+\cdots+N^2=\frac{N(N+1)(2N+1)}{6}.$$

Area Under $y = 1 - x^2$ over $[0, 1]$

We have

$$
\begin{aligned}
&\lim_{n\to\infty} \text{lower estimate}(n) \\
=&\lim_{n\to\infty} \left[n \cdot \frac{1}{n} - \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) \right] \\
=&\lim_{n\to\infty} \left[1 - \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} \right] \\
=&\lim_{n\to\infty} \left[1 - \frac{(1+\frac{1}{n})(2+\frac{1}{n})}{6} \right] \\
=&\frac{2}{3}
\end{aligned}
$$

Area Under $y = 1 - x^2$ over $[0, 1]$

Similarly,

$$
\begin{aligned}
&\lim_{n\to\infty} \text{upper estimate}(n) \\
=&\lim_{n\to\infty} \left[n \cdot \frac{1}{n} - \frac{1}{n^3} \left(0^2 + 1^2 + \cdots + (n-1)^2\right) \right] \\
=&\lim_{n\to\infty} \left[1 - \frac{1}{n^3} \frac{(n-1)n(2n-1)}{6} \right] \\
=&\lim_{n\to\infty} \left[1 - \frac{(1-\frac{1}{n})(2-\frac{1}{n})}{6} \right] \\
=&\frac{2}{3}
\end{aligned}
$$

Hence, the area under the graph of $y = 1 - x^2$ over $[0, 1]$ is $\dfrac{2}{3}$.

Note that the "area" under the graph of a function $y = 1 - x^2$ over the interval $[0, 1]$ may be used to represent other quantities.

For example, if we consider the velocity function of a particle moving along a line, say, $v(t) = 1 - t^2$, then all those rectangle areas computed in above example is

$$\underbrace{\left[1 - \left(\frac{k}{n}\right)^2\right]}_{velocity\ \ at\ \ t=k/n} \cdot \underbrace{\frac{1}{n}}_{time} < A_k < \left[1 - \left(\frac{k-1}{n}\right)^2\right] \cdot \frac{1}{n}$$

could be used to estimate the displacement of the particle during the time interval $\left[\frac{k-1}{n}, \frac{k}{n}\right]$. Hence the result of the limit calculation, $A = \frac{2}{3}$, is now the displacement of the particle during the time interval $0 \leq t \leq 1$.

Outline

Riemann Sums

The process in computing area in above example can obviously be applied to any continuous function f on the interval $[a, b]$.

The so called Riemann sum of a continuous function $f(x)$ on an interval $[a, b]$ with respect to a subdivision of the interval into n subintervals by the points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

is a straightforward generalization of rectangular approximation of area.

Riemann Sums

More precisely, denote the length of the i-th subinterval $[x_{i-1}, x_i]$ by Δx_i, and choose for each a point c_i in the subinterval $[x_{i-1}, x_i]$ for each $i = 1, 2, \ldots, n$. The corresponding Riemann sum is defined by:

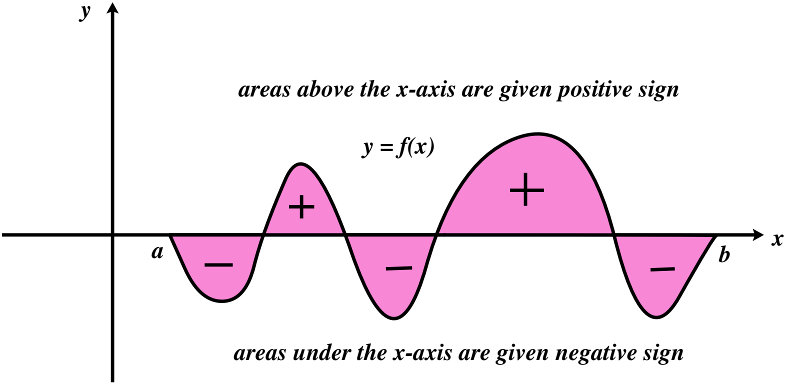
$$S_n = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

1. If $c_i = x_{i-1}$ for all i, then S_n is called a left (left point) Riemann sum.
2. If $c_i = x_i$ for all i, then S_n is called a right Riemann (right point) sum.
3. If $c_i = (x_{i-1} + x_i)/2$ for all i, then S_n is called a middle (middle point) Riemann sum.

For specific function, Riemann sums converge as the partition "gets finer and finer" (n gets larger and larger).

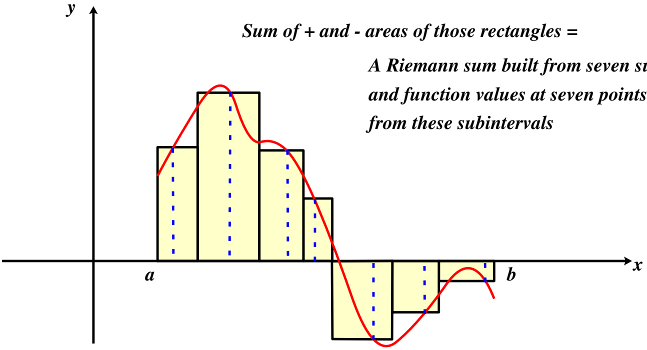
Riemann Sums and Signed Area

If you look at the graph of the function, a Riemann sum is just a rectangular approximation of the **signed area** (+ve/-ve area) between the graph and the x-axis, based on the chosen points x_i's and c_i's.



Riemann Sums and Signed Area

If you look at the graph of the function, a Riemann sum is just a rectangular approximation of the **signed area** (+ve/-ve area) between the graph and the x-axis, based on the chosen points x_i's and c_i's.



Riemann Sums and Definite Integrals

Taking into account the limiting behaviour of Riemann sums over finer and finer subdivisions, the definite integral of a continuous function $f(x)$ on an interval $[a, b]$ is defined and denoted by

$$\int_a^b f(x)dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i \qquad \text{whenever the limit exists.}$$

Geometrically speaking, if area above the x-axis is counted as positive area, and area below the x-axis as negative area, then

$$\int_a^b f(x)dx$$

is the sum of $+$ve and -ve area of the region between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$.

Riemann Sums and Definite Integrals

Just recall that the “rectangular areas” in the Riemann sum could actually mean certain quantity other than area, e.g., displacement.
The actually meaning of a definite integral

$$\int_a^b f(x)dx$$

in application relies on the meaning on the product $f(c_i)\Delta x_i$, i.e., the unit from “unit of the y-axis” times “unit of the x-axis”.

Riemann Sums and Definite Integrals

The summation notation, or the sigma notation, is often used to express the sum of a number of terms indexed by integers:

$$a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k$$

For example: $\displaystyle\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2.$

A basic property of the summation notation is: for any constants A, B,

$$\sum_{k=1}^{n} [Aa_k + Bb_k] = A \sum_{k=1}^{n} a_k + B \sum_{k=1}^{n} b_k.$$

Riemann Sums and Definite Integrals

The definite integral of a continuous function $f(x)$ on an interval $[a, b]$ can also be defined in a somewhat simplified but equivalent way, namely, by using subintervals of equal length

$$\Delta x = \frac{b-a}{n};$$

i.e., with subdivision points

$$a = x_0 < x_1 < x_2 < \cdots < x_i < \cdots x_n = b$$

where $x_i = x_0 + i\Delta x$, and c_i in $[x_{i-1}, x_i]$, such that

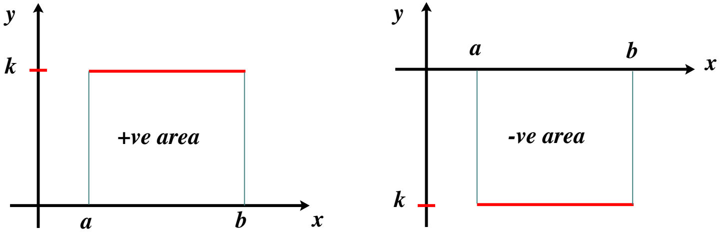
$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\frac{b-a}{n}$$ whenever the limit exists

Example: $\int_a^b k dx = k(b-a)$

An easy example: $f(x) = k$ where k is a constant.

Of course, by area consideration, we expect

$$\int_a^b f(x)dx = \int_a^b k dx = k(b-a)$$



Let's show that according to the Riemann sums.

Example: $\int_a^b kdx = k(b-a)$

Take any subdivision of the interval

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

we have a constant Riemann sum

$$k(x_1 - x_0) + k(x_2 - x_1) + k(x_3 - x_2) + \cdots + k(x_n - x_{n-1}) = k(x_n - x_0) = k(b-a)$$

since $f(c_i) = k$, no matter how you choose c_i in $[x_{i-1}, x_i]$.

The limit of the Riemann sums as $n \to \infty$ is then obviously $k(b-a)$.

Calculus IB: Lecture 20

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Outline

Outline

Riemann Sums and Definite Integrals

The definite integral of a continuous function $f(x)$ on an interval $[a, b]$ can be defined by using subintervals of equal length

$$\Delta x = \frac{b-a}{n};$$

i.e., with subdivision points

$$a = x_0 < x_1 < x_2 < \cdots < x_i < \cdots x_n = b$$

where $x_i = x_0 + i\Delta x$, and c_i in $[x_{i-1}, x_i]$, such that

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\frac{b-a}{n} \qquad \text{whenever the limit exists.}$$

The definite integral exists means the above limit exists on real number and its value does not depend on the choice of c_i.

Riemann Sums and Definite Integrals

In this section, we suppose all of definite integrals exist if we use notation

$$\int_a^b f(x)dx.$$

The existence of definite integral is not required in our course. We will give a brief sketch for this topic in next week.

Example: $\int_0^1 x dx = \frac{1}{2}$

Consider the partition of $[0, 1]$ into n subintervals by the subdivision points

$$0 < \frac{1}{n} < \frac{2}{n} < \frac{3}{n} < \cdots < \frac{n}{n} = 1.$$

We have the left-endpoint Riemann sum is

$$\begin{aligned}
&0 \cdot \frac{1}{n} + \frac{1}{n} \cdot \frac{1}{n} + \frac{2}{n} \cdot \frac{1}{n} + \cdots + \frac{n-1}{n} \cdot \frac{1}{n} \\
=&\frac{1}{n^2} \cdot [0 + 1 + 2 + 3 + \cdots + (n-1)] \\
=&\frac{1}{n^2} \cdot \frac{(n-1)n}{2} = \frac{1 - \frac{1}{n}}{2}
\end{aligned}$$

which tends to the limit $\frac{1}{2}$ as $n \to +\infty$.

Example: $\int_0^1 x dx = \frac{1}{2}$

Similarly, we have the right-endpoint Riemann sum is

$$\begin{aligned}
&\frac{1}{n} \cdot \frac{1}{n} + \frac{2}{n} \cdot \frac{1}{n} + \cdots + \frac{n}{n} \cdot \frac{1}{n} \\
=&\frac{1}{n^2} \cdot [1 + 2 + 3 + \cdots + n] \\
=&\frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{1 + \frac{1}{n}}{2}
\end{aligned}$$

which also tends to the limit $\frac{1}{2}$ as $n \to +\infty$.

Example: $\int_0^1 x dx = \frac{1}{2}$

Note that any other Riemann sum

$$S_n = \frac{1}{n}[c_1 + c_2 + \cdots + c_n]$$

with respect to the partition of the interval above, and c_i in $\left[\frac{i-1}{n}, \frac{i}{n}\right]$, is squeezed between the left-endpoint and right-endpoint Riemann sums

$$\frac{1}{2}\left(1 - \frac{1}{n}\right) \le S_n \le \frac{1}{2}\left(1 + \frac{1}{n}\right)$$

since the thin rectangular area with height $f(c_i) = c_i$ over the subinterval $\left[\frac{i-1}{n}, \frac{i}{n}\right]$ is squeezed by

$$\frac{1}{n} \cdot \frac{i-1}{n} \le \frac{1}{n} \cdot c_i \le \frac{1}{n} \cdot \frac{i}{n}.$$

Example: $\int_0^1 x dx = \frac{1}{2}$

Summing over the thin rectangular areas

$$\frac{1}{n} \cdot \frac{i-1}{n} \leq \frac{1}{n} \cdot c_i \leq \frac{1}{n} \cdot \frac{i}{n},$$

We have

$$\lim_{n \to \infty} \frac{1}{2}\left(1 - \frac{1}{n}\right) \leq \lim_{n \to \infty} \frac{1}{n}\left[c_1 + c_2 + \cdots + c_n\right] \leq \lim_{n \to \infty} \frac{1}{2}\left(1 + \frac{1}{n}\right),$$

then

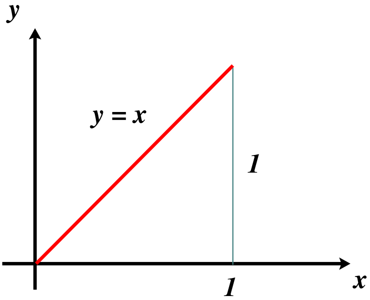
$$\int_0^1 x dx = \lim_{n \to \infty} \frac{1}{n}\left[c_1 + c_2 + \cdots + c_n\right] = \frac{1}{2}.$$

Example: $\int_0^1 x dx = \frac{1}{2}$

If we let x be time and velocity of a partical is be $v = f(x) = x$, then the definite integral (corresponds to the area under the curve)

$$\int_0^1 f(x) dx = \frac{1}{2}$$

is the displacement from time 0 to 1.



Example: $\int_0^1 e^x dx$

Find the value of the definite integral

$$\int_0^1 e^x dx$$

by working with left-endpoint Riemann sums.

We consider the subdivision points are: $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{k}{n}, \ldots, \frac{n}{n} = 1$.

Computing the function values at $0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}$, the left-endpoint Riemann sum S_n is

$$\begin{aligned} S_n =& \frac{1}{n}\left[e^0 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}}\right] \\ =& \frac{1}{n}\left[1 + e^{\frac{1}{n}} + \left(e^{\frac{1}{n}}\right)^2 + \cdots + \left(e^{\frac{1}{n}}\right)^{n-1}\right] \end{aligned}$$

Example: $\int_0^1 e^x dx$

The left-endpoint Riemann sum S_n is

$$
\begin{aligned}
S_n =& \frac{1}{n}\left[e^0 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + \cdots + e^{\frac{n-1}{n}}\right] \\
=& \frac{1}{n}\left[1 + e^{\frac{1}{n}} + \left(e^{\frac{1}{n}}\right)^2 + \cdots + \left(e^{\frac{1}{n}}\right)^{n-1}\right]
\end{aligned}
$$

Using the formula $1 + t + t^2 + \cdots + t^{n-1} = \frac{t^n - 1}{t-1}$ for any $t \neq 1$, we have

$$
\int_0^1 e^x dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{n} \frac{\left(e^{\frac{1}{n}}\right)^n - 1}{e^{\frac{1}{n}} - 1} = \frac{e-1}{\lim\limits_{n\to\infty} \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}}} = e - 1
$$

where we use the result $\lim\limits_{n\to\infty} \dfrac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} = 1.$

Example: $\int_0^1 e^x dx$

Then we check the limit $\lim_{n\to\infty} \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} = 1$.

Let $x = \frac{1}{n}$, then $n \to \infty$ means $x \to 0$ and we have

$$\lim_{n\to\infty} \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} = \lim_{x\to 0} \frac{e^x - 1}{x} = \lim_{x\to 0} \frac{e^x}{1} = 1,$$

where we use L'Hôpital's rule (check the conditions!).

Example: Area of a Circle $= \pi r^2$

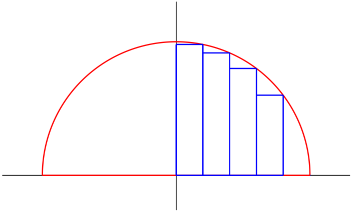
In geometry, the area enclosed by a circle of radius r is πr^2. How to explain this result in the view of Riemann sums/definite integral?

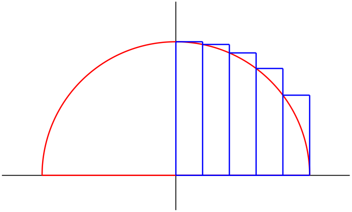
Example: Area of a Circle $= \pi r^2$

Based on the symmetric property of circle, we only needs to consider the right-top part of the circle.

We start by subdividing the interval $[0, 1]$ into n subintervals of the same length $\frac{1}{n}$ by the subdivision points

$$0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n}{n} = 1$$



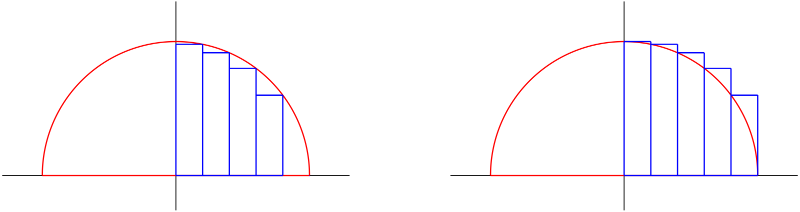


Example: Area of a Circle $= \pi r^2$

The equation of a circle with radius r and center origin is $x^2 + y^2 = r^2$. Hence, we have $y = \sqrt{r^2 - x^2}$.

Over the interval $\left[\frac{(k-1)r}{n}, \frac{kr}{n}\right]$, $k = 1, \ldots, n$, we have the area sandwich:

$$\frac{1}{n} \cdot \sqrt{r^2 - \left(\frac{k}{n}\right)^2 \cdot r^2} < A_k < \frac{1}{n} \cdot \sqrt{r^2 - \left(\frac{k-1}{n}\right)^2 \cdot r^2}$$



Example: Area of a Circle $= \pi r^2$

Following the previous trick, we should compute the limits

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sqrt{r^2 - \left(\frac{k}{n}\right)^2 \cdot r^2}$$

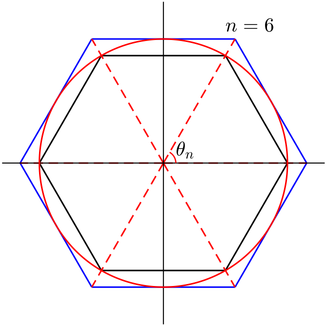
or

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sqrt{r^2 - \left(\frac{k-1}{n}\right)^2 \cdot r^2}$$

How to compute these limits?

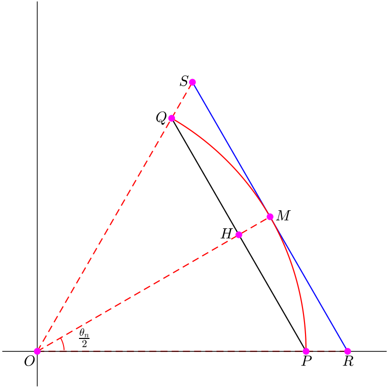
Example: Area of a Circle $= \pi r^2$

We can use triangles to establish the sandwich. Consider the area enclosed by the blue and black polygons when $n \to \infty$ and show that the area of a circle with radius r is πr^2.



Our analysis cannot use the result that the area of circular sector is $\frac{1}{2}r^2\theta_n$, since it is based on the area of the circle is πr^2.

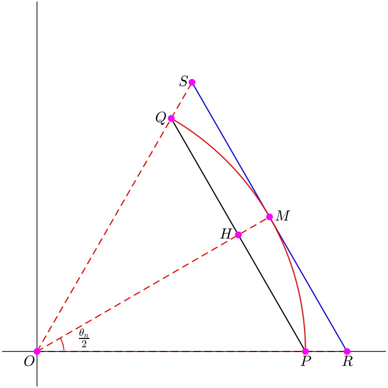
Example: Area of a Circle $= \pi r^2$



$$\text{Area of the circle } = n \cdot \text{Area(Sector } OPMQ)$$

$$\text{Area}(\triangle OPQ) < \text{Area(Sector } OPMQ) < \text{Area}(\triangle ORS)$$

Example: Area of a Circle $= \pi r^2$



$$OP = OM = OQ = r$$

$$OH = r \cos \frac{\theta_n}{2}$$

$$PQ = 2PH = 2r \sin \frac{\theta_n}{2}$$

$$MR = MS = OM \tan \frac{\theta_n}{2} = r \tan \frac{\theta_n}{2}$$

$$RS = 2MR = 2r \tan \frac{\theta_n}{2}$$

$$\text{Area}(\triangle OPQ) = \frac{1}{2} \cdot PQ \cdot OH = r^2 \sin \frac{\theta_n}{2} \cos \frac{\theta_n}{2}$$

$$\text{Area}(\triangle ORS) = \frac{1}{2} \cdot RS \cdot OM = r^2 \tan \frac{\theta_n}{2}$$

Example: Area of a Circle $= \pi r^2$

Let $A_k = \text{Area}(\text{Sector } OPMQ)$, then the area of the circle is

$$S = \sum_{k=1}^{n} A_k = A_1 + A_2, \ldots, A_n.$$

Hence, we have

$$r^2 \sin\frac{\theta_n}{2}\cos\frac{\theta_n}{2} < A_k < r^2 \tan\frac{\theta_n}{2}$$

and

$$n \cdot r^2 \sin\frac{\theta_n}{2}\cos\frac{\theta_n}{2} < S < n \cdot r^2 \tan\frac{\theta_n}{2}.$$

Taking $n \to \infty$, we can obtain $S = \frac{1}{2}\pi r^2$ by sandwich theorem.

Example: Area of a Circle $= \pi r^2$

Let $x = \dfrac{\pi}{n} \to 0^+$. Since $\theta_n = \dfrac{2\pi}{n}$, we have

$$
\begin{aligned}
&\lim_{n\to\infty} n \cdot r^2 \cdot \sin\frac{\theta_n}{2} \cdot \cos\frac{\theta_n}{2} \\
=&\lim_{n\to\infty} n \cdot r^2 \cdot \sin\frac{\pi}{n} \cdot \cos\frac{\pi}{n} \\
=&\lim_{n\to\infty} \pi r^2 \cdot \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} \cdot \cos\frac{\pi}{n} \\
=&\pi r^2 \cdot \lim_{n\to\infty} \cos\frac{\pi}{n} \cdot \lim_{n\to\infty} \frac{\sin\frac{\pi}{n}}{\frac{\pi}{n}} \\
=&\pi r^2 \cdot \lim_{x\to 0^+} \cos x \cdot \lim_{x\to 0} \frac{\sin x}{x} \\
=&\pi r^2
\end{aligned}
$$

Example: Area of a Circle $= \pi r^2$

Since $x = \dfrac{\pi}{n} \to 0^+$ and $\theta_n = \dfrac{2\pi}{n}$, we have

$$
\begin{aligned}
& \lim_{n\to\infty} n \cdot r^2 \cdot \tan \frac{\theta_n}{2} \\
= & \lim_{n\to\infty} n \cdot r^2 \cdot \frac{\sin \frac{\pi}{n}}{\cos \frac{\pi}{n}} \\
= & \lim_{x\to 0} r^2 \cdot \frac{\pi}{x} \cdot \frac{\sin x}{\cos x} \\
= & \pi r^2 \cdot \lim_{x\to 0^+} \frac{\sin x}{x} \cdot \lim_{x\to 0^+} \cos x \\
= & \pi r^2
\end{aligned}
$$

Hence, the sandwich theorem means $S = \pi r^2$.

Example: Area of a Circle $= \pi r^2$

Note that our analysis use the identity

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \tag{1}$$

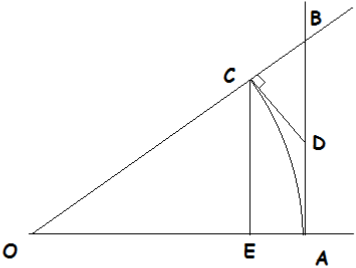
The proof of (1) in Lecture 07 uses

$$\text{Area(circular sector)} = \frac{1}{2} r^2 \theta \tag{2}$$

to show $\sin\theta < \theta < \tan\theta$ when $0 < \theta < \dfrac{\pi}{2}$.

However, the formula (2) comes from Area(circle) $= \pi r^2$, which leads to circular argument!!! We desire another proof of $\sin\theta < \theta < \tan\theta$ without using the identity of area.

Show $\sin\theta < \theta$ without Area

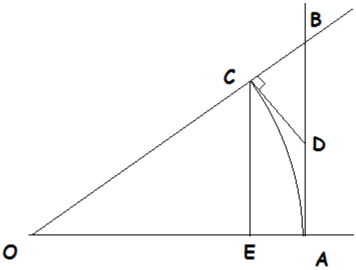


Let $\theta = \angle AOC$ be the corresponding angle of the sector and the lengths of OC and OA is the radius r. Then the length of arc AC is $r\theta$.

The shortest distance from point C to line AO is $CE = r\sin\theta$, where CE is orthogonal to OA. Another path from point C to line OA is arc CA, which is longer than CE because it is not the shortest path. So we have

$$r\sin\theta < r\theta \Longrightarrow \sin\theta < \theta.$$

Show $\theta < \tan\theta$ without Area



Since the set of points bound by sector OCA is a subset of the set of points bound by quadrilateral $OCDA$, the perimeter of quadrilateral $OCDA$ must be longer than the perimeter of the sector OCA (we use both of them are convex sets). Hence, we have $CD + DA >$ arc $CA = r\theta$ and

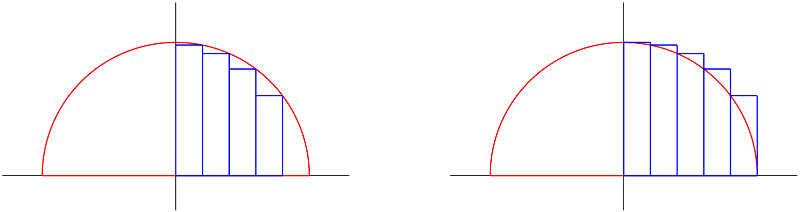
$$r\tan\theta = BA = BD + DA > +CD + DA > r\theta \implies \theta < \tan\theta.$$

Example: Area of a Circle $= \pi r^2$

The equation of a circle with radius r and center origin is $x^2 + y^2 = r^2$. Hence, the curve of upper semicircle corresponds to $y = \sqrt{r^2 - x^2}$ and we have

$$\text{area of quarter circle} = \int_0^r \sqrt{r^2 - x^2}dx = \frac{\pi r^2}{4}$$

$$\text{area of semicircle} = \int_{-r}^r \sqrt{r^2 - x^2}dx = \frac{\pi r^2}{2}$$



Outline

Fundamental Theorem of Calculus

Calculating definite integrals by the original limit definition based on Riemann sums is very difficult in general.

We have seen that it is so complicated even if we only want to derive the area formula of a circle.

Sometimes, there is an easier way compute definite integrals in general.

Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)

Let f be a continuous function on the closed interval $[a, b]$. If $F(x)$ is an antiderivative of f, i.e., $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a),$$

which is often denoted as $F(x)\Big|_a^b$ or $\Big[F(x)\Big]_a^b$.

In other words, whenever you can find

$$\int f(x)dx = F(x) + C,$$

it is just one step further to find the corresponding definite integral:

$$\int_a^b f(x)dx = F(b) - F(a).$$

Fundamental Theorem of Calculus

Example

Evaluate $\displaystyle\int_0^1 x dx$.

Since $\dfrac{d}{dx}\left(\dfrac{x^2}{2}\right) = x$, we have by the fundamental theorem of calculus

$$\int_0^1 x dx = \frac{1}{2}x^2 \Big|_0^1 = \frac{1}{2}(1)^2 - \frac{1}{2}(0)^2 = \frac{1}{2}.$$

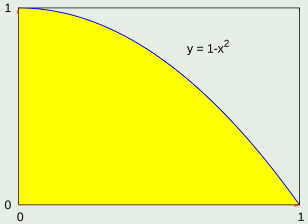
Fundamental Theorem of Calculus

Example

Evaluate $\int_0^1 (1-x^2)dx$.

Using the fundamental theorem of calculus, we have

$$\int_0^1 (1-x^2)dx = \left[x - \frac{1}{3}x^3\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}.$$



Fundamental Theorem of Calculus

Example

Find the area under the graph of the function over the given interval:

1. $f(x) = x^2$, $1 \le x \le 2$.

$$\int_1^2 x^2 dx = \left[\frac{1}{3}x^3\right]_1^2 = \frac{1}{3}(2)^3 - \frac{1}{3}(1)^3 = \frac{7}{3}$$

2. $g(x) = e^x$, $0 \le x \le 3$.

$$\int_0^3 e^x dx = \left[e^x\right]_0^3 = e^3 - e^0 = e^3 - 1$$

Fundamental Theorem of Calculus

Example

Find the area under the graph of the function over the given interval:

1. $h(x) = \sin x$, $0 \leq x \leq \pi$.

$$\int_0^{\pi} \sin x dx = \Big[-\cos x\Big]_0^{\pi} = [-\cos(\pi)] - [-\cos 0] = 2$$

2. $u(x) = \cos x$, $0 \leq x \leq \pi/2$.

$$\int_0^{\pi/2} \cos x dx = \Big[\sin x\Big]_0^{\pi/2} = \sin\frac{\pi}{2} - \sin 0 = 1$$

Fundamental Theorem of Calculus

Example

A few more definite integrals:

- $\displaystyle\int_1^2 (5x^4 - 6x^2 + x - 1)dx = \left[x^5 - 2x^3 + \frac{x^2}{2} - x\right]_1^2$
  $\displaystyle= [32 - 16 + 2 - 2] - [1 - 2 + \frac{1}{2} - 1] = \frac{35}{2}$
- $\displaystyle\int_0^2 (5e^x - 3x^2)dx = \left[5e^x - x^3\right]_0^3 = [5e^3 - 27] - [5e^0 - 0] = 5e^3 - 32$
- $\displaystyle\int_1^5 \frac{1}{x}dx = \left[\ln x\right]_1^5 = \ln 5 - \ln 1 = \ln 5$

Calculus IB: Lecture 21

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Outline

Outline

Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)

Let f be a continuous function on the closed interval $[a, b]$. If $F(x)$ is an antiderivative of f, i.e., $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a),$$

which is often denoted as $F(x)\Big|_a^b$ or $\Big[F(x)\Big]_a^b$.

In other words, whenever you can find

$$\int f(x)dx = F(x) + C,$$

it is just one step further to find the corresponding definite integral:

$$\int_a^b f(x)dx = F(b) - F(a).$$

The Proof of Fundamental Theorem of Calculus

Partition the interval $[a, b]$ with n subintervals of equal length $\dfrac{b-a}{n}$.

By the mean value theorem Theorem, there exists x^* in $[x_{i-1}, x_i]$ such that

$$F(x_i) - F(x_{i-1}) = F'(x_i^*)(x_i - x_{i-1}) = f(x_i^*) \cdot \frac{b-a}{n}$$

Then, we take the Riemann sum of the function f on $[a, b]$:

$$\sum_{i=1}^{n} f(x_i^*) \cdot \frac{b-a}{n} = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(x_n) - F(x_0) = F(b) - F(a).$$

Thus by the definition of definite integral, we have

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) \cdot \frac{b-a}{n} = F(b) - F(a).$$

Example of Fundamental Theorem of Calculus

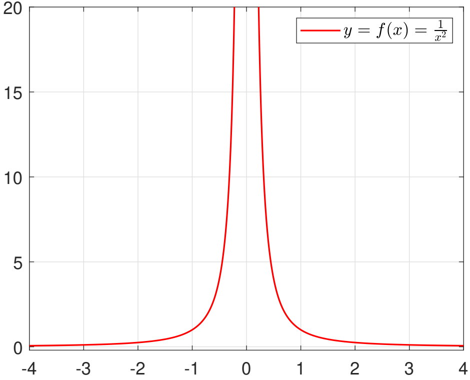
Example

Find the definite integral $\int_{-1}^{2} \frac{1}{x^2} dx$

Since $\left(-\frac{1}{x}\right)' = \frac{1}{x^2}$, we use fundamental theorem of calculus to obatin

$$\int_{-1}^{2} \frac{1}{x^2} dx = \left(-\frac{1}{x}\right)\Big|_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}$$

Example of Fundamental Theorem of Calculus



The definite integral $\int_{-1}^{2} \frac{1}{x^2} dx$ can NOT be negative!!!

Outline

Fundamental Theorem of Calculus

Theorem (Fundamental Theorem of Calculus)

Let f be a continuous function on the closed interval $[a, b]$. If $F(x)$ is an antiderivative of f, i.e., $F'(x) = f(x)$, then

$$\int_a^b f(x)dx = F(b) - F(a),$$

which is often denoted as $F(x)\big|_a^b$ or $\big[F(x)\big]_a^b$.

The function $f(x) = \dfrac{1}{x^2}$ is NOT continuous at 0 and $\displaystyle\int_{-1}^{2} \frac{1}{x^2}dx \neq -\frac{3}{2}$.

In fact, if one uses Riemann sum to approximate this integral, the limit of the Riemann sum is ∞.

Riemann Sums and Integrability

The definite integral of a continuous function $f(x)$ on an interval $[a, b]$ can be defined by using subintervals of equal length

$$\Delta x = \frac{b-a}{n};$$

i.e., with subdivision points $a = x_0 < x_1 < x_2 < \cdots < x_i < \cdots x_n = b$, where $x_i = x_0 + i\Delta x$, and c_i in $[x_{i-1}, x_i]$.

If the limit of Riemann sum

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \cdot \Delta$$

exists on real numbers, we say the function f is Riemann integrable if the limit of the Riemann sum exists and has a unique limit L. The limit is called the definite integral of f from a to b, denoted by

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i) \cdot \Delta = L$$

Integrable Functions

Theorem (sufficient condition)

If f is continuous on on $[a, b]$, then f must be (Riemann) integrable.

Theorem (sufficient condition$^+$)

If f is continuous over $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuous points, then f is integrable on $[a, b]$.

Theorem (necessary condition)

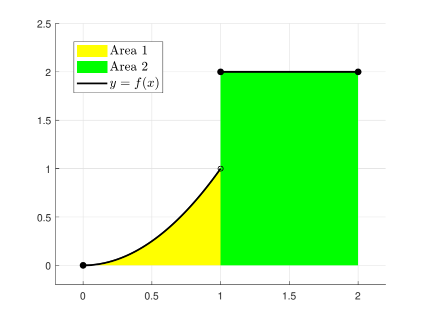
If f is (Riemann) integrable on $[a, b]$, then f must be bounded on $[a, b]$.

Example

The function

$$f(x) = \begin{cases} x^2 & 0 \le x < 1 \\ 2 & 1 \le x \le 2 \end{cases}$$

is integrable om $[0, 2]$.



The definite integral of

$$f(x) = \begin{cases} x^2 & 0 \le x < 1 \\ 2 & 1 \le x \le 2 \end{cases}$$

on $[0, 2]$ is

$$\int_0^2 f(x)dx = \lim_{t \to 1^-} \int_0^t f(x)dx + \int_1^2 f(x)dx$$

Non-Integrable Functions

Theorem (necessary condition)

If f is (Riemann) integrable on $[a, b]$, then f must be bounded on $[a, b]$.

Example

The function $f(x) = \dfrac{1}{x^2}$ is unbounded on $[-1, 2]$.

Even if $M > 0$ is sufficient large, we have $x_M = \dfrac{1}{\sqrt{M+1}}$ in $[-1, 2]$

such that $f(x_M) = M + 1 > M$.

Non-Integrable Functions

Theorem (sufficient condition$^+$)

If f is continuous over $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuous points, then f is integrable on $[a, b]$.

Example (Dirichlet Function)

Dirichlet function is defined as follows

$$D(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number,} \\ 0 & \text{if } x \text{ is not a rational number.} \end{cases}$$

Dirichlet function is nowhere continuous (in other words, there are infinite number of discontinuous points) and not (Riemann) integrable on any $[a, b]$ when $a < b$.

Non-Integrable Functions

Consider the behavior of Dirichlet function

$$D(x) = \begin{cases} 1 & \text{if } x \text{ is a rational number,} \\ 0 & \text{if } x \text{ is a irrational number.} \end{cases}$$

on interval $[a, b]$, where $a < b$.

Let $a = 0$ and $b = 1$. given rational number 0.123, we can construct infinite irrational numbers

$$0.123x_1x_2x_3 \dots x_n \dots$$

Since each x_i can be select from $0, 1, \dots, 9$, we can think

$$\frac{\#\text{rational numbers}}{\#\text{irrational numbers}} \approx \lim_{n\to\infty} \left(\frac{1}{10}\right)^n = 0$$

Intuitively, rationals number in $[0, 1]$ is much less than irrational number.

Non-Integrable Functions

In other words, $D(x) = 0$ almost everywhere. If we think the area of a segment is 0, then it is reasonable that the "area" of "graph" under $D(x)$ on interval $[a, b]$ is also 0.

In fact, we can define other types of integration (not Riemann integration) to characterize the area under the graph of a function.

Non-Integrable Functions

For example, in the view of Lebesgue integration, Dirichlet function is integrable on $[a, b]$ and

$$\int_a^b D(x)dx = 0.$$

However, the expression

$$\int_a^b D(x)dx$$

is undefined by Riemann integration.

Integrable Functions

In homework and exam of MATH 1013, "integrable" always refers to Riemann integrable.

Some Properties of Integrable Functions

Let f and g are integrable on closed interval $[a, b]$, then

1. for any constants A, B, we have
$$\int_a^b [Af(x) + Bg(x)]dx = A\int_a^b f(x)dx + B\int_a^b f(x)dx$$
2. for any constants $a \leq b \leq c$, we have
$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$
3. if $f(x) \geq g(x)$ on $[a, b]$, then
$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$
4. if $a > b$, we define $\int_b^a f(x)dx = -\int_a^b f(x)dx$ in conventional.

Outline

Integral Sandwiches for $\sin x$ and $\cos x$

Consider that $\cos x \le 1$ and $\sin x < x$. Then for any $x \ge 0$,

$$\cos x \le 1 \Longrightarrow \int_0^x \cos t dt \le \int_0^x 1 dt \Longleftrightarrow \sin t \Big|_0^x \le t \Big|_0^x$$

$$\sin x \le x \Longrightarrow \int_0^x \sin t dt \le \int_0^x t dt \Longleftrightarrow -\cos t \Big|_0^x \le \frac{x^2}{2}$$

$$1 - \frac{x^2}{2} \le \cos x \Longrightarrow \int_0^x \left(1 - \frac{t^2}{2}\right) dt \le \int_0^x \cos t dt = \sin x$$

$$x - \frac{x^3}{3!} \le \sin x \Longrightarrow \int_0^x \left(t - \frac{t^3}{3!}\right) dt \le \int_0^x \sin t dt = -\cos x + 1$$

$$\cdots\cdots \Longrightarrow \cdots\cdots$$

Exclamation mark "!" means factorial, that is, $k! = 1 \cdot 2 \cdot \cdots \cdot k$ for any positive integer k. We define $0! = 1$ in conventional.

Integral Sandwiches for $\sin x$ and $\cos x$

Repeating such procedures, we have (show that by induction)

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots - \frac{x^{4n-1}}{(4n-1)!} \leq \sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{x^{4n+1}}{(4n+1)!}$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots - \frac{x^{2n-2}}{(2n-2)!} \leq \cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{x^{2n+2}}{(2n+2)!}$$

We can approximate $\sin x$ and $\cos x$ by polynomial functions

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + \frac{x^{4n+1}}{(4n+1)!}$$

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + \frac{x^{2n+2}}{(2n+2)!}$$

We can use the polynomial

$$p(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

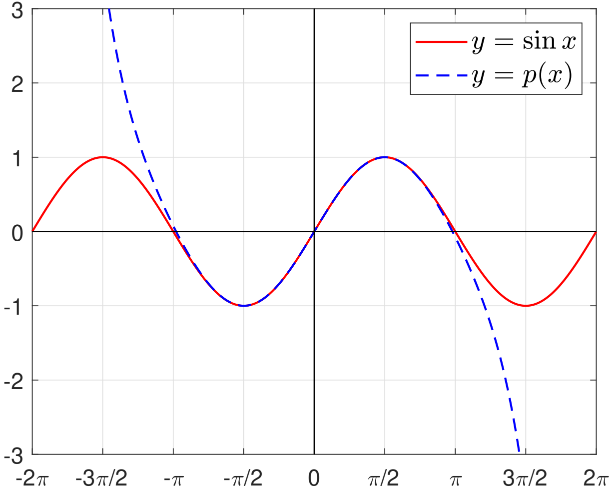
to estimate the function value of $\sin x$, then we have

$$p(x) \leq \sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \Longrightarrow \sin x - p(x) \leq \frac{x^9}{9!}$$

If we restrict x on $\left[0, \frac{\pi}{4}\right]$, the above inequalities implies

$$0 \leq \sin x - p(x) \leq \frac{x^9}{9!} \leq \frac{\left(\frac{\pi}{4}\right)^9}{9!} = 3.13 \times 10^{-7}$$

Integral Sandwiches for $\sin x$ and $\cos x$



Integral Sandwiches for $\sin x$ and $\cos x$

We can use the polynomial

$$q(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

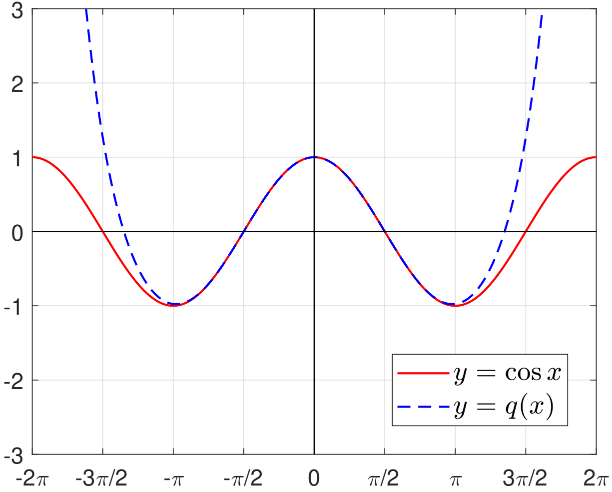
to estimate the function value of $\cos x$, then we have

$$q(x) \leq \cos x \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \Longrightarrow 0 \leq \cos x - q(x) \leq \frac{x^8}{8!}$$

If we restrict x on $\left[0, \frac{\pi}{4}\right]$, the above inequalities implies

$$0 \leq \cos x - q(x) \leq \frac{x^8}{8!} \leq \frac{\left(\frac{\pi}{4}\right)^8}{8!} = 3.59 \times 10^{-6}$$

Integral Sandwiches for $\sin x$ and $\cos x$



Taylor Series and Linear Approximation

More general, for $f(x)$ that is infinitely differentiable, we can approximate it by Taylor series

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

In above examples, we take $a = 0$ and $f(x)$ be $\sin x / \cos x$.

We can also think this strategy is an extension of linear approximation

$$f(x) \approx f(a) + f'(a)(x-a)$$

Taylor Series and Optimization

Let $a = x_k$, we have

$$f(x) \approx f(x_k) + f'(x_k)(x - x_k) + \frac{f''(x_k)}{2}(x - x_k)^2 + \frac{f'''(x_k)}{3!}(x - x_k)^3 + \cdots$$

The iteration of gradient descent is

$$\begin{aligned} x_{k+1} =& x_k - \frac{1}{L} f'(x_k) \\ =& \arg\min_x \left[f(x_k) + f'(x_k)(x - x_k) + \frac{L}{2}(x - x_k)^2 \right] \end{aligned}$$

Recall that we suppose $f''(x) \leq L$ for positive L in the analysis of convex optimization, which means the update is optimizing and upper bound of first three terms in Taylor series.

It is easy to check

$$\begin{aligned} x_{k+1} =& x_k - \frac{1}{L} f'(x_k) \\ =& \arg\min_x \left[f(x_k) + f'(x_k)(x - x_k) + \frac{L}{2}(x - x_k)^2. \right] \end{aligned}$$

We define

$$g(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{L}{2}(x - x_k)^2,$$

then $g''(x) = L > 0$ and $g'(x) = f'(x_k) + L(x - x_k)$.

Hence, $g(x)$ is convex and x_{k+1} is its unique critical point (minimizer).

Taylor Series and Optimization

What happens if we directly minimize the first three terms?

$$f(x) \approx f(x_k) + f'(x_k)(x - x_k) + \frac{f''(x_k)}{2}(x - x_k)^2 + \frac{f'''(x_k)}{3!}(x - x_k)^3 + \cdots$$

We define $h(x) = f(x_k) + f'(x_k)(x - x_k) + \dfrac{f''(x)}{2}(x - x_k)^2$, then $h''(x) = f''(x)$ and $h'(x) = f'(x_k) + f''(x_k)(x - x_k)$.

Hence, if $f(x)$ is strictly-convex then $h''(x) = f''(x_k) > 0$ is convex and x_{k+1} is its unique critical point (minimizer) of $h(x)$ is

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)},$$

which leads to Newton's Method!

Taylor Series and Optimization

In theoretical, we can also optimize

$$l(x) = f(x_k) + f'(x_k)(x - x_k) + \frac{f''(x_k)}{2}(x - x_k)^2 + \frac{f'''(x_k)}{3!}(x - x_k)^3$$

to establish an optimization algorithm, but solving such sub-problem is more complicated and difficult to be extended to high-dimensional case.

Taylor Series and Optimization

Similar to linear approximation, the approximation

$$f(x) \approx f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

has high accuracy when x is close to a.

Intuitively, if x is far away from a, we require a larger n to increase $n!$ and control the magnitude of

$$\frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Calculus IB: Lecture 22

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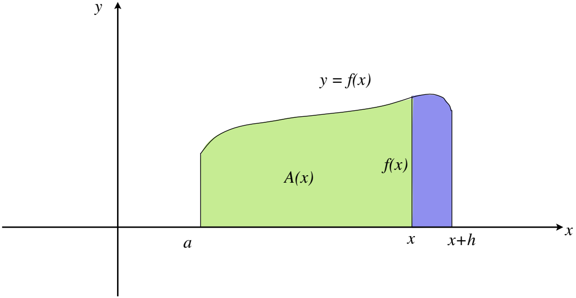
Outline

Outline

Fundamental Theorem of Calculus (v2)

In the geometric view of fundamental theorem of calculus, we consider the following "area function" defined by

$$A(x) = \int_a^x f(t)dt.$$

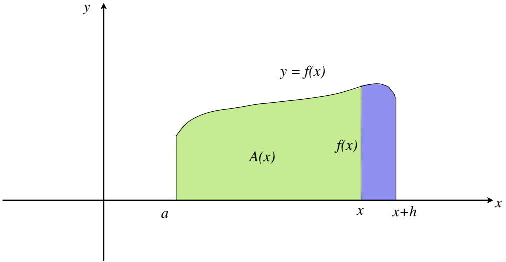


Fundamental Theorem of Calculus (v2)

In the geometric view of fundamental theorem of calculus, we consider the following “area function” defined by

$$A(x) = \int_a^x f(t)dt.$$

Then we study the derivative of $A(x)$.



Fundamental Theorem of Calculus (v2)

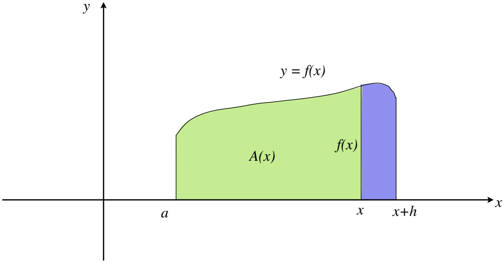
The "area function" is

$$A(x) = \int_a^x f(t)dt.$$

Note that given $h \approx 0$, then

$$A(x+h) - A(x) = \int_x^{x+h} f(t)dt \approx f(x)h$$

and hence we may expect: $A'(x) = \lim_{h \to 0} \dfrac{A(x+h) - A(x)}{h} = f(x)$



Fundamental Theorem of Calculus (v2)

To be precise, just consider the "area sandwich" for $h > 0$:

$$\min_{x \le t \le x+h} f(t)h \le A(x+h) - A(x) \le \max_{x \le t \le x+h} f(t)h$$

As f is continuous on $[a, b]$, we have by taking limits

$$\begin{aligned} f(x) &= \lim_{h \to 0^+} \min_{x \le t \le x+h} f(t) \\ &\le \lim_{h \to 0^+} \frac{A(x+h) - A(x)}{h} \\ &\le \lim_{h \to 0^+} \max_{x \le t \le x+h} f(t) = f(x) \end{aligned}$$

For $h < 0$, we consider the interval $[x + h, x]$ and end up with

$$\lim_{h \to 0^-} \frac{A(x+h) - A(x)}{h} = f(x)$$

Fundamental Theorem of Calculus (v2)

In other word, $A'(x) = f(x)$ and hence the area function $A(x)$ is an antiderivative of $f(x)$. Rewrite this as a theorem, we have:

Theorem (Fundamental Theorem of Calculus v2)

Let f be a continuous function on the interval $[a, b]$. Then

$$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$

Fundamental Theorem of Calculus (v2)

Recall that the antiderivative of f can be expressed as

$$\int f(x)dx = F(x) + C,$$

and there must be a constant C such that (fundamental theorem of calculus v2)

$$\int_a^x f(t)dt = F(x) + C$$

Putting in $x = a$, we have

$$0 = \int_a^a f(t)dt = F(a) + C \quad \Longleftrightarrow \quad C = -F(a)$$

Therefore we have

$$\int_a^x f(t)dt = F(x) - F(a)$$

which corresponds to previous version of fundamental theorem of calculus

$$\int_a^b f(t)dt = F(b) - F(a)$$

Fundamental Theorem of Calculus (v2)

Example

Let $y = \int_1^{x^2} \sin 3t dt$, find $\frac{dy}{dx}$

Let $u = x^2$ and use chain rule, then

$$\frac{dy}{dx} = \left(\frac{d}{du}\int_1^u \sin 3t dt\right) \cdot \frac{du}{dx}$$
$$= (\sin 3u) \cdot \frac{du}{dx} = (\sin 3x^2) \cdot \frac{dx^2}{dx} = 2x \sin 3x^2$$

Exercise

Instead of using the fundamental theorem of calculus directly, try to use the limit definition of derivative to find derivatives above.

Outline

1. Fundamental Theorem of Calculus (v2)
2. Net Change Theorem
3. Substitution Rules in Definite Integral

Net Change Theorem

Just by rewriting the fundamental theorem of calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

where $F'(x) = f(x)$ into another form, we have net change theorem

$$\int_a^b F'(x)dx = F(b) - F(a)$$

since $F(b) - F(a)$ is the change in $y = F(x)$ when x changes from a to b.

Net Change Theorem

Example

A particle moves along a line with velocity function $v(t) = t^2 - 2t$ (meters per second). Find the displacement and distance traveled during the time interval $1 \le t \le 6$.

The displacement is

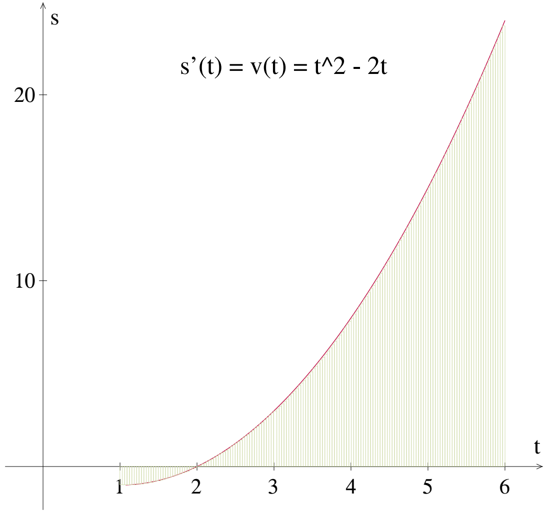
$$s(6) - s(1) = \int_1^6 (t^2 - 2t)dt = \left[t^3/3 - t^2\right]_1^6 = (72 - 36) - (1/3 - 1) = \frac{110}{3}$$

Note that the speed function is $|v(t)| = |t^2 - 2t|$. Hence the distance traveled is

$$\begin{aligned}\int_1^6 |t(t-2)|dt &= \int_1^2 -(t^2 - 2t)dt + \int_2^6 (t^2 - 2t)dt \\ &= \left[-t^3/3 + t^2\right]_1^2 + \left[t^3/3 - t^2\right]_2^6 = 38,\end{aligned}$$

where 2 is the point the sign of $t(t-2)$ changes.

Geometric view of above example:



Example

Suppose a volcano spewed out solid materials is of the rate $r(t)$ at which are given in the following table.

t(in seconds)	0	1	2	3	4	5	6
$r(t)$(tones per second)	2	10	24	36	46	54	60

(a) Give upper and lower estimates for the total quantity $Q(6)$ of erupted materials after 6 seconds.

(b) Use the Midpoint Rule (Midpoint Riemann Sum) to estimate $Q(6)$.

(a) Using the table,

$$2+10+24+36+46+54 < \int_0^6 r(t)dt < 10+24+36+46+54+60$$

$$172 < Q(6) < 230 \text{ (tones)}$$

(b) Using three subintervals of length 2, with subdivision points $0 < 2 < 4 < 6$, we have midpoints $1, 3, 5$ and hence

$$Q(6) \approx 2(10+36+54) = 200 \text{ (tones)}$$

Outline

The Substitution Rule

Theorem (The Substitution Rule in Indefinite Integral)

If $u = g(x)$ is a differentiable function whose range is an interval I, and $f(x)$ is continuous on I, then (since $u = g(x)$ means $du = g'(x)dx$)

$$\int f(g(x))g'(x)dx = \int f(u)du.$$

Theorem (The Substitution Rule in Definite Integral)

If $u = g(x)$ is a differentiable function whose range is an interval I, and $f(x)$ is continuous on I, then

$$\int_a^b f(g(x))g'(x)dx \overset{u=g(x)}{=\!=} \int_{g(a)}^{g(b)} f(u)du.$$

The Substitution Rule

Example

Find $\int_0^2 \sqrt{4x+1}dx$

Let $u = g(x) = 4x+1$, then $g(0) = 1$, $g(2) = 9$, $\sqrt{4x+1} = u^{\frac{1}{2}}$ and

$$\int_0^2 \sqrt{4x+1}dx \overset{u=4x+1}{=} \int_1^9 \frac{1}{4}u^{\frac{1}{2}}du = \frac{1}{4}\left[\frac{2}{3}u^{3/2}\right]_1^9 = \frac{1}{6}(27-1) = \frac{13}{3}$$

Example

Find $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x dx$

We hope the expression only depends on $\cos x$ or $\sin x$:

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x dx = \int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \ (\sin x \ dx)$$
$$= \int_0^{\frac{\pi}{2}} (1 - \cos^2 x) \cos^4 x \ (-d \cos x)$$

Let $u = g(x) = \cos x$, then $g(0) = \cos 0 = 1$, $g(1) = \sin \frac{\pi}{2} = 0$ and

$$\int_0^{\frac{\pi}{2}} \sin^3 x \cos^4 x dx = -\int_{g(0)}^{g(\frac{\pi}{2})} (1 - u^2) u^4 du$$
$$= -\left[\frac{t^5}{5} - \frac{t^7}{7}\right]_1^0 = -\left[0 - \left(\frac{1}{5} - \frac{1}{7}\right)\right] = \frac{2}{35}$$

The Substitution Rule

In above example, we use the fact

$$\int_{g(0)}^{g(\frac{\pi}{2})} (1-u^2)u^4 du = \int_1^0 (1-u^2)u^4 du$$

where $g(x) = \cos x$.

The range of the definite integration should follow by $g(\frac{\pi}{2}) = 0$ and $g(0) = 1$. The expression allows $1 > 0$.

The Substitution Rule

Theorem (The Substitution Rule in Definite Integral)

If $u = g(x)$ is a differentiable function whose range is an interval I, and $f(x)$ is continuous on I, then

$$\int_a^b f(g(x))g'(x)dx \overset{u=g(x)}{=} \int_{g(a)}^{g(b)} f(u)du.$$

If we want to find

$$\int_{u_1}^{u_2} f(u)du,$$

we can also use the substitution rule by take $u = g(x)$ and compute

$$\int_{x_1}^{x_2} f(g(x))g'(x)dx,$$

where $x_1 = g^{-1}(u_1)$ and $x_2 = g^{-1}(u_2)$ (suppose the inverse function of g exists in the interval).

The Substitution Rule

The expression

$$\int_{x_1}^{x_2} f(g(x))g'(x)dx$$

looks more complicated than

$$\int_{u_1}^{u_2} f(u)du.$$

However, such substitution could be very useful in some specific problem.

The Substitution Rule

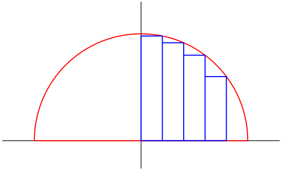
Recall that the area of quarter circle with radius r is

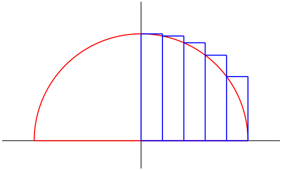
$$\frac{1}{4}\pi r^2 = \int_0^r \sqrt{r^2 - x^2}dx$$

It is difficult to verify this result by Riemann sum:

$$\lim_{n\to\infty} \frac{r}{n} \sum_{k=1}^{n} \sqrt{r^2 - \left(\frac{k}{n}\right)^2 \cdot r^2}$$

However, we can use substitution rule to find this definite integral.





The Substitution Rule

Let $x = r\sin\theta$. Since the function $\sin\theta$ is a one-to-one function in $[0, \frac{\pi}{2}]$ and $r\sin 0 = 0$, $r\sin\frac{\pi}{2} = r$, we have

$$\begin{aligned}
\int_0^r \sqrt{r^2 - x^2}dx &= \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2\theta}\; d(r\sin\theta) \\
&= \int_0^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2\theta} \cdot r\cos\theta d\theta \\
&= r^2 \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2\theta}\cos\theta d\theta \\
&= r^2 \int_0^{\frac{\pi}{2}} \cos^2\theta d\theta \\
&= r^2 \int_0^{\frac{\pi}{2}} \frac{1 + \cos 2\theta}{2} d\theta \\
&= r^2 \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}} = r^2\left[\frac{\pi}{4} + \frac{\sin\pi}{4} - \left(0 + \frac{\sin 0}{4}\right)\right] = \frac{1}{4}\pi r^2
\end{aligned}$$

The Substitution Rule

Since the circle is symmetric, the result

$$\frac{1}{4}\pi r^2 = \int_0^r \sqrt{r^2 - x^2}dx$$

means the area of semi-circle is

$$\frac{1}{2}\pi r^2 = \int_{-r}^r \sqrt{r^2 - x^2}dx$$

and the area of the circle is πr^2.

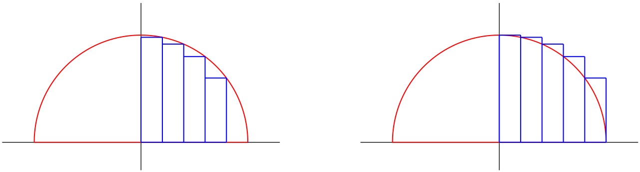
Symmetry in Definite Integrals

Recall that the integrand $f(x) = r\sqrt{r^2 - x^2}$ in above example is an even function, that is $f(x) = f(-x)$ and the interval of the definite integral

$$\int_{-r}^{r} \sqrt{r^2 - x^2}dx$$

is also symmetric with respect to 0. Hence

$$\int_{-r}^{r} \sqrt{r^2 - x^2}dx = 2\int_{0}^{r} \sqrt{r^2 - x^2}dx = 2\int_{-r}^{0} \sqrt{r^2 - x^2}dx$$



Consider a "difficult" problem

$$\int_{-1}^{1} \frac{\tan x}{1 + 3x^2 + 5x^4 + 7x^6} = ?$$

- Applying Riemann sum is very complicated.
- Using fundamental theorem of calculus is also very difficult.

Is this problem real "difficult"?

Symmetry in Definite Integrals

We define

$$f(x) = \frac{\tan x}{1 + 3x^2 + 5x^4 + 7x^6}.$$

In fact, $f(x)$ is an odd function, that is

$$f(x) = -f(-x).$$

Recall that definite integral is the signed area between the graph of the function and the x-axis.

Odd function means the graph is symmetric with respect to origin.

Since the range $[-1, 1]$ also is symmetric and $f(x)$ is well defined on it, we must have

$$\int_{-1}^{1} f(x)dx = 0$$

Symmetry in Definite Integrals

Since the signed areas are canceled, we must have

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1} \frac{\tan x}{1+3x^2+5x^4+7x^6} dx = 0.$$