Multivariate Statistical Analysis

Lecture 13

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Outline

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The Conjugate Prior for the Covariance

Theorem

If $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, n)$ *and* $\mathbf{\Sigma}$ *has a prior distribution* $\mathcal{W}^{-1}(\mathbf{\Psi}, m)$*, then the conditional distribution of* $\mathbf{\Sigma}$ *given* $\mathbf{A}$ *is the inverted Wishart distribution*

$$\mathcal{W}^{-1}(\mathbf{A} + \mathbf{\Psi}, n + m).$$

Let each of $\mathbf{x}_1, \ldots, \mathbf{x}_N$ has distribution $\mathcal{N}_p(\mathbf{0}, \mathbf{\Sigma})$ independently and $n = N - 1$, then the sample covariance

$$\mathbf{S} = \frac{1}{n}\sum_{i=1}^{n}(\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top \sim \mathcal{W}_p(\mathbf{\Sigma}, n).$$

If $\mathbf{\Sigma} \sim \mathcal{W}_p^{-1}(\mathbf{\Psi}, m)$, then we have

$$\mathbf{\Sigma} \,|\, \mathbf{S} \sim \mathcal{W}^{-1}(n\mathbf{S} + \mathbf{\Psi}, n + m).$$

The Inverted Wishart Distribution

Theorem

Let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ be observations from $\mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Suppose $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ have prior densities

$$n\left(\boldsymbol{\mu} \;\middle|\; \boldsymbol{\nu}, \frac{\boldsymbol{\Sigma}}{K}\right) \qquad \textit{and} \qquad w^{-1}(\boldsymbol{\Sigma} \mid \boldsymbol{\Psi}, m)$$

respectively, where $n = N - 1$. Then the posterior density of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ given

$$\bar{\mathbf{x}} = \frac{1}{N}\sum_{\alpha=1}^{N} \mathbf{x}_\alpha \quad \textit{and} \quad \mathbf{S} = \frac{1}{N-1}\sum_{\alpha=1}^{N} (\mathbf{x}_\alpha - \bar{\mathbf{x}})(\mathbf{x}_\alpha - \bar{\mathbf{x}})^\top$$

is

$$n\left(\boldsymbol{\mu} \;\middle|\; \frac{N\bar{\mathbf{x}} + K\boldsymbol{\nu}}{N+K}, \frac{\boldsymbol{\Sigma}}{N+K}\right) \cdot w^{-1}\left(\boldsymbol{\Sigma} \mid \boldsymbol{\Psi} + n\mathbf{S} + \frac{NK(\bar{\mathbf{x}} - \boldsymbol{\nu})(\bar{\mathbf{x}} - \boldsymbol{\nu})^\top}{N+K}, N + m\right).$$

Outline

The Characteristic Function of Wishart Distribution

Theorem

Let $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, n)$*, then the characteristic function of*

$$a_{11}, a_{22}, \ldots, a_{pp}, 2a_{12}, \ldots, 2a_{p-1,p},$$

is is given by

$$\mathbb{E}\left[\exp(\mathrm{i}\,\mathrm{tr}(\mathbf{A\Theta}))\right] = \left(\det\left(\mathbf{I} - 2\mathrm{i}\mathbf{\Theta\Sigma}\right)\right)^{-\frac{n}{2}},$$

where $\mathbf{\Theta} \in \mathbb{R}^{p\times p}$ *is symmetric.*

Outline

The density of F-distribution with m and n degrees of freedom in univariate case is

$$\frac{1}{B\left(\frac{m}{2},\frac{n}{2}\right)}\left(\frac{m}{n}\right)^{\frac{n}{2}} u^{\frac{n}{2}-1}\left(1+\frac{m}{n}\cdot u\right)^{-\frac{m+n}{2}},$$

where

$$B\left(\frac{m}{2},\frac{n}{2}\right)=\frac{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}{\Gamma\left(\frac{m+n}{2}\right)}.$$

How to generalized it to multivariate case?

Matrix F-Distribution

Let $\mathbf{A} \sim \mathcal{W}_p(\mathbf{I}, n)$ and $\mathbf{B} \sim \mathcal{W}_p(\mathbf{\Sigma}^{-1}, m)$ be independent, then

$$\mathbf{U} = \mathbf{B}^{-1/2}\mathbf{A}\mathbf{B}^{-1/2},$$

has matrix F-distribution with n and m degrees of freedom.

Its density function is

$$f(\mathbf{U}) = \frac{\Gamma_p\left(\frac{m+n}{2}\right)(\det(\mathbf{\Sigma}))^{-\frac{n}{2}}}{\Gamma_p(\frac{m}{2})\Gamma_p(\frac{n}{2})} \cdot (\det(\mathbf{U}))^{\frac{n-p-1}{2}} \left(\det(\mathbf{I} + \mathbf{U}\mathbf{\Sigma}^{-1})\right)^{-\frac{m+n}{2}}.$$

It is natural to define the multivariate Beta function as

$$B_p(a, b) = \frac{\Gamma_p(a)\Gamma_p(b)}{\Gamma_p(a+b)}.$$

Matrix Beta Distribution

The density of Beta distribution with parameters $m/2$ and $n/2$ in univariate case is

$$f(w) = \frac{1}{B(\frac{m}{2}, \frac{n}{2})} \cdot w^{\frac{n}{2}-1}(1-w)^{\frac{m}{2}-1},$$

where

$$B\left(\frac{m}{2}, \frac{n}{2}\right) = \frac{\Gamma(\frac{m}{2})\Gamma(\frac{n}{2})}{\Gamma\left(\frac{m+n}{2}\right)}.$$

How to generalized it to multivariate case?

Matrix Beta Distribution

Let $\mathbf{A} \sim \mathcal{W}_p(\mathbf{\Sigma}, n)$ and $\mathbf{B} \sim \mathcal{W}_p(\mathbf{\Sigma}, m)$ be independent, then

$$\mathbf{W} = (\mathbf{A} + \mathbf{B})^{-1/2}\mathbf{A}(\mathbf{A} + \mathbf{B})^{-1/2}$$

has matrix Beta distribution with parameters $n/2$ and $m/2$ if $\mathbf{0} \prec \mathbf{W} \prec \mathbf{I}$ and 0 elsewhere.

Its density function is

$$f(\mathbf{W}) = \frac{1}{B_p(\frac{n}{2}, \frac{m}{2})} \cdot (\det(\mathbf{W}))^{\frac{n-p-1}{2}} (\det(\mathbf{I} - \mathbf{W}))^{\frac{m-p-1}{2}},$$

which does not depend on $\mathbf{\Sigma}$.