Multivariate Statistical Analysis

Lecture 04

Fudan University

luoluo@fudan.edu.cn

Outline

1. Zeroth-Order Optimization
2. Gaussian Smoothing
3. Complexity Analysis

Optimization Problems: Your Feeling Before This Class

Settings	Smooth Convex	Nonsmooth Convex	Smooth Nonconvex	Nonsmooth Nonconvex
1st/2nd				
0th				

Settings	Smooth Convex	Nonsmooth Convex	Smooth Nonconvex	Nonsmooth Nonconvex
1st/2nd				
0th				

Multivariate statistics is all you need.

Outline

Zeroth-Order Optimization

In real applications, the explicit expression of gradient may be hard to achieve.

1. Black-box attack to DNN.
2. Simulation optimization.

Zeroth-Order Optimization

We consider the problem

$$\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}),$$

where $f:\mathbb{R}^d \to \mathbb{R}$ is continuous.

We focus on the scheme

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta_t \cdot \frac{f(\mathbf{x}_t + \delta\mathbf{u}_t) - f(\mathbf{x}_t)}{\delta} \cdot \mathbf{u}_t$$

for some $\eta_t > 0$ and $\delta > 0$, where $\mathbf{u}_t \sim \mathcal{N}_d(\mathbf{0}, \mathbf{I})$.

Outline

1 Zeroth-Order Optimization

2 Gaussian Smoothing

3 Complexity Analysis

Gaussian Smoothing

We define the Gaussian smoothing of $f(\cdot)$ as

$$f_\delta(\mathbf{x}) = \mathbb{E}[f(\mathbf{x}+\delta\mathbf{u})] = \int_{\mathbb{R}^d} \frac{1}{(2\pi)^{d/2}} f(\mathbf{x}+\delta\mathbf{u}) \exp\left(-\frac{1}{2}\|\mathbf{u}\|_2^2\right) \mathrm{d}\mathbf{u}$$

for some $\delta > 0$, where $\mathbf{u} \sim \mathcal{N}_d(\mathbf{0}, \mathbf{I})$

The continuity of $f(\cdot)$ means $f_\delta(\cdot)$ is differentiable and it holds

$$\nabla f_\delta(\mathbf{x}) = \mathbb{E}\left[\frac{f(\mathbf{x}+\delta\mathbf{u}) - f(\mathbf{x})}{\delta} \cdot \mathbf{u}\right].$$

1. If $f(\cdot)$ is G-Lipschitz continuous, then

$$|f_\delta(\mathbf{x}) - f(\mathbf{x})| \leq \delta G\sqrt{d}.$$

2. If $f(\cdot)$ is L-smooth, then

$$|f_\delta(\mathbf{x}) - f(\mathbf{x})| \leq \frac{L\delta^2 d}{2} \quad \text{and} \quad \|\nabla f_\delta(\mathbf{x}) - \nabla f(\mathbf{x})\|_2^2 \leq \frac{L\delta(d+3)^{3/2}}{2}.$$

Gaussian Smoothing

The properties of Gaussian smoothing:

1. If $f(\cdot)$ is G-Lipschitz continuous, then $f_\delta(\cdot)$ is G-Lipschitz continuous and $G\sqrt{d}/\delta$-smooth.
2. If $f(\cdot)$ is L-smooth, then $f_\delta(\cdot)$ is L-smooth.
3. If $f(\cdot)$ is convex, then $f_\delta(\cdot)$ is convex and $f_\delta(\cdot) \geq f(\cdot)$.

Zeroth-Order Optimization

We study the scheme

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta_t \mathbf{g}_\delta(\mathbf{x}_t; \mathbf{u}_t),$$

where

$$\mathbf{g}_\delta(\mathbf{x}; \mathbf{u}) = \frac{f(\mathbf{x} + \delta \mathbf{u}) - f(\mathbf{x})}{\delta} \cdot \mathbf{u}.$$

1. If $f(\cdot)$ is G-Lipschitz continuous, then

$$\mathbb{E}\left\|\mathbf{g}_\delta(\mathbf{x}; \mathbf{u})\right\|_2^2 \leq G^2(d+4)^2.$$

2. If $f(\cdot)$ is L-smooth, then

$$\mathbb{E}\left\|\mathbf{g}_\delta(\mathbf{x}; \mathbf{u})\right\|_2^2 \leq \frac{L^2\delta^2(d+6)^3}{2} + 2(d+4)\left\|\nabla f(\mathbf{x})\right\|_2^2.$$

Outline

Theorem (Nonsmooth Convex)

Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is convex and G-Lipschitz. The iteration

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta_t \mathbf{g}_\delta(\mathbf{x}_t; \mathbf{u}_t)$$

holds that

$$\begin{aligned}
&\frac{1}{\sum_{t=0}^{T-1} \eta_t} \sum_{t=0}^{T-1} \eta_t \mathbb{E}[(f(\mathbf{x}_t) - f(\mathbf{x}^*)] \\
\leq& \delta G \sqrt{d} + \frac{1}{2\sum_{t=0}^{T-1} \eta_t} \left(\|\mathbf{x}_0 - \mathbf{x}^*\|_2^2 + G^2 (d+4)^2 \sum_{t=0}^{T-1} \eta_t^2 \right).
\end{aligned}$$

Zeroth-Order Optimization

Theorem (Smooth Convex)

Suppose $f : \mathbb{R}^d \to \mathbb{R}$ is convex and L-smooth. The iteration

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta_t \mathbf{g}_\delta(\mathbf{x}_t; \mathbf{u}_t)$$

with $\eta_t = 1/(4L(d+4))$ holds that

$$\frac{1}{T}\sum_{t=0}^{T-1}(f(\mathbf{x}_t) - f(\mathbf{x}^*)) \leq \frac{4L(d+4)\,\|\mathbf{x}_0 - \mathbf{x}^*\|_2^2}{T} + \frac{9L\delta^2(d+4)^2}{25}.$$

Additionally suppose $f(\cdot)$ is μ-strongly convex, then

$$\mathbb{E}\left[\|\mathbf{x}_T - \mathbf{x}^*\|_2^2 - \Delta\right] \leq \left(1 - \frac{\mu}{8L(d+4)}\right)^T \left(\|\mathbf{x}_0 - \mathbf{x}^*\|_2^2 - \Delta\right),$$

where $\Delta = \dfrac{18\delta^2 L(d+4)^2}{25\mu}$.

Zeroth-Order Optimization

The differentiability of $\nabla f_\delta(\cdot)$ and the fact

$$\mathbb{E}[\mathbf{g}_\delta(\mathbf{x};\mathbf{u})] = \nabla f_\delta(\mathbf{x})$$

means the mini-batch version scheme

$$\mathbf{x}_{t+1} = \mathbf{x}_t - \eta_t \cdot \frac{1}{b}\sum_{i=1}^{b} \mathbf{g}_\delta(\mathbf{x}_t;\mathbf{u}_{t,i})$$

can reduce the iteration numbers.